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TOWARD A GENERAL THEORY OF ELASTIC EQUILIBRIUM EQUATIONS FOR AN ISOTROPIC BODY

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TOWARD A GENERAL THEORY OF ELASTIC EQUILIBRIUM EQUATIONS FOR AN ISOTROPIC BODY

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ABSTRACT. Elastic equilibrium problems are formulated and solved in the form of definite integrals wherever possible. No mass forces are assumed. Problems in which the boundary forces and boundary displacements are specified, as well as the general mixed boundary condition problems, are treated. Infinite planes and spheres are considered as boundaries. The problems reduce to ordinary or partial differential equations.

INTRODUCTION

The methods of a general nature, which have previously been used to solve problems of elastic equilibrium for an isotropic body may be reduced substantially to two -- Lamé's classical method for series expansions of *simple functions* and that for definite integrals, commonly called the Betti-Cerruti method. The first, which was really created and successfully employed to solve many other problems in mechanics and mathematical physics, cannot be applied beyond a very restricted number of cases as far as our problem is concerned. In addition, it is complicated by the not easily surmountable difficulty of determining the constants when we satisfy the surface conditions. The second method has indeed an appearance of great generality, but -- perhaps precisely because of this great generality -- it has all the characteristics of an abstract method, showing itself to be little adapted to, or flexible in, the relatively simple problems of the equilibrium of isotropic bodies. Although notable results have been obtained by its use, it is my opinion that these results are to be attributed more to the study and thought which the eminent practitioners of the science -- who wanted to give this method life -- have put into it, rather than to the intrinsic value of the method. /129*

I turned these considerations over in my mind many times when I had to delve into the subject for another purpose. From my study I was able to derive general principles which seem to me more suitable than those heretofore in use to obtain, or at least to attempt, a solution of elastic equilibrium problems for isotropic bodies. Exposition of these principles and application of them to various special problems will be the subject of this report, and of any other which may follow it. /130

I hope that when I have demonstrated the fact that all the problems whose solution is known may be solved by a uniform method, simply, and even perhaps, elegantly, and the fact that another large class of problems is also susceptible of relatively simple solution, my views and my work will be judged with a certain indulgence.

As far as is possible for me to do so, I will present the solutions of the

* Numbers in the margin indicate pagination in the original foreign text.

separate problems in the form of definite integrals, since these solutions have the advantage--over those presented in the form of series expansions--that they include all the elements of the problem in an artificial way: initial data and results. Besides, methods of verification are often quick and simple. We may also obtain analytical expressions in series form from the analytical expressions for the definite integrals, whereas the inverse problem is not so simple.

In this Report, I will deal with problems in which the surface of the elastic body is a plane or a sphere.

I. Equations and General Principles

1. Let us immediately decide to call x, y, z the coordinates of any point in space and--every time that an attempt is made to represent a definite integral function--to call ξ, η, ζ the points of the variable point on the surface, or in the portion of the space over which integration is extended. Let us also agree always to indicate by S the portion of space which is occupied by the elastic body which is finite, or infinite, and connected, and by σ its external surface, which we will assume in every case to satisfy the conditions under which Green's theorem may be applied in space S.

In order not to introduce useless complications, we will always assume that the elastic body is not subject to external mass forces. Then the indefinite equations of elastic equilibrium of a homogeneous and isotropic body may be given in either of the two following forms /131

$$\left.\begin{aligned}
&\Delta^2 u + \frac{\lambda+\mu}{\mu}\frac{\partial\theta}{\partial x} = 0,\\
&\Delta^2 v + \frac{\lambda+\mu}{\mu}\frac{\partial\theta}{\partial y} = 0, \quad \Delta^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}, \quad \theta = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z};\\
&\Delta^2 w + \frac{\lambda+\mu}{\mu}\frac{\partial\theta}{\partial z} = 0,
\end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned}
&(\lambda+2\mu)\frac{\partial\theta}{\partial x} + 2\mu\left(\frac{\partial\varpi_2}{\partial z}-\frac{\partial\varpi_3}{\partial y}\right) = 0, \quad \varpi_1 = \frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),\\
&(\lambda+2\mu)\frac{\partial\theta}{\partial y} + 2\mu\left(\frac{\partial\varpi_3}{\partial x}-\frac{\partial\varpi_1}{\partial z}\right) = 0, \quad \varpi_2 = \frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right),\\
&(\lambda+2\mu)\frac{\partial\theta}{\partial z} + 2\mu\left(\frac{\partial\varpi_1}{\partial y}-\frac{\partial\varpi_2}{\partial x}\right) = 0, \quad \varpi_3 = \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right),
\end{aligned}\right\} \tag{2}$$

in which λ and μ are Lamé's two known constants, which, as is known, are subject to conditions

$$3\lambda + 2\mu > 0, \qquad \mu > 0. \tag{3}$$

The components X_n, Y_n, Z_n of the stress acting over one surface element of the elastic body located in the position of normal n are given by the following formulas

$$X_n = \lambda\theta\cos nx + 2\mu\left(\frac{du}{dn} + \varpi_3\cos ny - \varpi_2\cos nz\right), \tag{4}$$

$$\left.\begin{aligned} Y_n &= \lambda\theta\cos ny + 2\mu\left(\frac{dv}{dn} + \varpi_1\cos nz - \varpi_3\cos nx\right), \\ Z_n &= \lambda\theta\cos nz + 2\mu\left(\frac{dw}{dn} + \varpi_2\cos nx - \varpi_1\cos ny\right), \\ \frac{d}{dn} &= \frac{\partial}{\partial x}\cos nx + \frac{\partial}{\partial y}\cos ny + \frac{\partial}{\partial z}\cos nz. \end{aligned}\right\} \tag{4}$$

If, for purposes of brevity, we indicate by L, M, N, the values assumed by $-X_n$, $-Y_n$, $-Z_n$ in the points of σ, when normal n to σ is understood to be directed to the interior of S, the most general problem, which we wish to treat here, may be stated thus: *Let us find a system of functions u, v, w which are regular and satisfy expression (1) or expression (2) in S, such that three of the expressions u, v, w; L, M, N,--of which only two, say, u and L, corresponding to the same coordinate axis--assume assigned values in points of σ.*

2. To solve these problems let us begin by establishing certain fundamental formulas. Therefore, let us indicate by G the ordinary Green function relative to space S and to point (x, y, z) inside S, which, as is known, when considered as a function of the coordinates of the variable point (ξ, η, ζ) or of the coordinates (x, y, z) of the pole, is regular and harmonic* in S, except when $\xi=x$, $\eta=y$, $\zeta=z$, in which case it becomes infinite in the manner of /132

$$\frac{1}{r}, \quad r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2},$$

and vanishes in the points of σ. Let us similarly indicate by G_1 the other Green function which within S satisfies the same conditions as G, while on σ the normal derivative $\frac{dG_1}{dn}$ assumes a constant value** which is zero only in the case in which S extends to infinity. In this last case, G and G_1 become zero at infinity in the manner of potential functions. It is known that under these conditions, functions G and G_1 exist and are uniquely determined in the most general cases.

If we then find that -- when θ is a harmonic function in S -- the first

* To speak more precisely let us say that by harmonic function we mean any function whose second-order differential parameter is generally zero, and by harmonic and regular function we mean any function that, besides being harmonic, is uniform, finite, and continuous, together with the first-order derivatives in that region of space in which we consider it.

** If we want to construct function G_1 effectively, it is perhaps more convenient to start from Klein's definition, by which G_1 has in S two poles of first order at the points (x, y, z), (x_0, y_0, z_0) instead of one, with the residues +1 and -1, and such that $\frac{dG_1}{dn}$ become zero over σ.

equation (1) may be written as

$$\Delta^2\left[u+\frac{\lambda+\mu}{2\mu}x\,\theta\right]=0,$$

by applying Green's theorem to the functions G and $u + \frac{\lambda+\mu}{2\mu} x\theta$ in S, it is found, assuming that u and θ are regular in S, that

$$u=\frac{1}{4\pi}\int_\sigma u\frac{dG}{dn}d\sigma-\frac{\lambda+\mu}{2\mu}x\,\theta+\frac{\lambda+\mu}{8\pi\mu}\int_\sigma \xi\,\theta\frac{dG}{dn}d\sigma \tag{5}$$

a formula which may also be written

$$u=\frac{1}{4\pi}\int_\sigma u\frac{dG}{dn}d\sigma+\frac{\lambda+\mu}{8\pi\mu}\int_\sigma (\xi-x)\,\theta\frac{dG}{dn}d\sigma. \tag{5'}$$ /133

Similarly, under the same conditions for u and θ, application of Green's theorem in S to the functions G_1 and $u+\frac{\lambda+\mu}{2\mu}x\theta$ gives the following

$$u=-\frac{1}{4\pi}\int_\sigma \frac{du}{dn}G_1\,d\sigma-\frac{\lambda+\mu}{2\mu}x\,\theta-\frac{\lambda+\mu}{8\pi\mu}\int_\sigma \frac{d\,\xi\,\theta}{dn}G_1\,d\sigma+\text{cost.}, \tag{6}$$

a formula which may also be written

$$u=-\frac{1}{4\pi}\int_\sigma \frac{du}{dn}G_1\,d\sigma-\frac{\lambda+\mu}{8\pi\mu}\int_\sigma \frac{d[(\xi-x)\,\theta]}{dn}G_1\,d\sigma+\text{cost.} \tag{6'}$$

Similar considerations also hold true, of course, for the other two equations (1).

Here we should like again to observe that, if S extends to infinity, the constants which appear in the second terms of expressions (6) or (6') are zero. However, in this case we will assume for the applicability of Green's theorem that function $u + \frac{\lambda+\mu}{2\mu} x\,\theta$ and those like it become zero at infinity with order higher than $\frac{1}{r}$.

3. If the attempt is now made to solve the problem of elastic equilibrium when the values of displacements u, v, w on the surface σ are given, we may observe that, because of expression (5) and similar ones, we can write

$$u=\frac{1}{4\pi}\int_\sigma u\frac{dG}{dn}d\sigma-\frac{\lambda+\mu}{2\mu}x\,\theta+\frac{\lambda+\mu}{8\pi\mu}\int_\sigma \xi\,\theta\frac{dG}{dn}d\sigma, \quad \Big\} \tag{5''}$$

$$\left.\begin{aligned} v &= \frac{1}{4\pi}\int_\sigma v \frac{dG}{dn} d\sigma - \frac{\lambda+\mu}{2\mu} y\theta + \frac{\lambda+\mu}{8\pi\mu}\int_\sigma \eta \vartheta \frac{dG}{dn} d\sigma, \\ w &= \frac{1}{4\pi}\int_\sigma w \frac{dG}{dn} d\sigma - \frac{\lambda+\mu}{2\mu} z\theta + \frac{\lambda+\mu}{8\pi\mu}\int_\sigma \zeta \vartheta \frac{dG}{dn} d\sigma \end{aligned}\right\} \qquad (5'')$$

and that in these formulas the first terms on the right side are known. The problem is then reduced to determining function θ, harmonic and regular in S, so that the equation

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

or /134

$$\left.\begin{aligned} \frac{\lambda+3\mu}{2\mu}\theta = \frac{1}{4\pi}\left[\frac{\partial}{\partial x}\int_\sigma u \frac{dG}{dn} d\sigma + \frac{\partial}{\partial y}\int_\sigma v \frac{dG}{dn} d\sigma + \frac{\partial}{\partial z}\int_\sigma w \frac{dG}{dn} d\sigma\right] - \\ - \frac{\lambda+\mu}{2\mu}\left(x\frac{\partial\theta}{\partial x} + y\frac{\partial\theta}{\partial y} + z\frac{\partial\theta}{\partial z}\right) + \\ + \frac{\lambda+\mu}{8\pi\mu}\left[\frac{\partial}{\partial x}\int_\sigma \xi\theta \frac{dG}{dn} d\sigma + \frac{\partial}{\partial y}\int_\sigma \eta\theta \frac{dG}{dn} d\sigma + \frac{\partial}{\partial z}\int_\sigma \zeta\theta \frac{dG}{dn} d\sigma\right]. \end{aligned}\right\} \qquad (7)$$

will be identically satisfied.

It may be noted that, based on the hypothesis made regarding θ, the right and left sides in equation (7) are two harmonic functions; equation (7) will thus be identically satisfied in S if it is identically satisfied in all the points of surface σ. The problem which has been formulated may therefore be reduced to determining the values which θ must assume in the points of σ by means of the equation to which equation (7) is reduced in the points of σ. Harmonic function θ will be constructed with these values.

If instead an attempt is made to solve the elastic equilibrium problem when the values for L, M, N are given on σ, we will note that for expressions (4) we may write

$$\frac{du}{dn} = -\frac{1}{2\mu}L - \frac{\lambda}{2}\theta\cos nx - \varpi_3\cos ny + \varpi_2\cos nz,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

and that therefore for expression (6) and similar ones the following equations may be written:

$$\left.\begin{aligned} u = \frac{1}{8\pi\mu}\int_\sigma L G_1 d\sigma + \frac{1}{4\pi}\int_\sigma\left[\frac{\lambda}{2\mu}\theta\cos n\xi + \varpi_3\cos n\eta - \varpi_2\cos n\zeta\right]G_1 d\sigma - \\ - \frac{\lambda+\mu}{2\mu}x\theta - \frac{\lambda+\mu}{8\pi\mu}\int_\sigma \frac{d\xi\theta}{dn}G_1 d\sigma + \text{cost.}, \end{aligned}\right\} \qquad (8)$$

$$
\left.\begin{aligned}
v &= \frac{1}{8\pi\mu}\int_\sigma M G_1\, d\sigma + \frac{1}{4\pi}\int_\sigma \left[\frac{\lambda}{2\mu}\theta\cos n\eta + \varpi_1\cos n\zeta - \varpi_3\cos n\xi\right] G_1\, d\sigma - \\
&\qquad - \frac{\lambda+\mu}{2\pi}\, y\,\theta - \frac{\lambda+\mu}{8\pi\mu}\int_\sigma \frac{d\eta\,\theta}{dn}\, G_1\, d\sigma + \text{cost.}, \\
w &= \frac{1}{8\pi\mu}\int_\sigma N G_1\, d\sigma + \frac{1}{4\pi}\int_\sigma \left[\frac{\lambda}{2\mu}\theta\cos n\zeta + \varpi_2\cos n\xi - \varpi_1\cos n\eta\right] G_1\, d\sigma - \\
&\qquad - \frac{\lambda+\mu}{2\mu}\, z\,\theta - \frac{\lambda+\mu}{8\pi\mu}\int_\sigma \frac{d\zeta\,\theta}{dn}\, G_1\, d\sigma + \text{cost.}
\end{aligned}\right\} \tag{8}
$$

in which the first terms on the left sides are known. In this case, the problem is reduced to determining the four functions θ, $\overline{\omega}_1$, $\overline{\omega}_2$, $\overline{\omega}_3$, harmonic and regular in S, and the fact that there they identically satisfy the four equations /135

$$
\left.\begin{aligned}
\theta &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}; \quad 2\varpi_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \\
2\varpi_2 &= \frac{\partial u}{\partial z} - \frac{\partial v}{\partial x}, \quad 2\varpi_3 = \frac{\partial v}{\partial x} - \frac{\partial w}{\partial y}.
\end{aligned}\right\} \tag{9}
$$

We will not develop these equations, but we will not fail to observe that remarks may be made about them similar to those which we have made about expression (7).

If, finally, the attempt is then made to solve the problem of elastic equilibrium in the case in which several of the u, v, w; L, M, N values are given with the indicated restriction, expressions (5) and (8) will be of timely use to us, and the rest of the solution will proceed as in the preceding case.

Determination of harmonic function θ from expression (7) or of functions θ; $\overline{\omega}_1$, $\overline{\omega}_2$, $\overline{\omega}_3$ from expression (9) or from similar equations constitutes the peculiar difficulty of the corresponding problem of elastic equilibrium. We are not concerned with demonstrating the fact here that expression (7) or equations (9), for example, keep their meaning on surface σ, and that they are suitable for determining the values there of θ or of θ; $\overline{\omega}_1$, $\overline{\omega}_2$, $\overline{\omega}_3$ as finite and continuous functions of the points in σ. That will certainly be the case under very general conditions, but an exact reply to such questions would entail a theorem for the solution of our problems, and we hope to be able to return to this later on.

II. Problems in Which the Elastic Body Is Limited by an Infinite Plane

1. *Case in which u, v, w are given on the limiting plane.* Let us assume that the elastic body is limited by the plane z = 0 and occupies that region of space in which z > 0. In this case Green's function G for the point (x, y, z) inside S reduces to

$$\frac{1}{r} - \frac{1}{r_1}$$

where r and r_1 are the distances of point (x, y z) and of the point symmetrical to it with respect to plane z = 0 from the same point (ξ,η,ζ) in S. Hence noting that for ζ = 0 /136

$$\frac{dG}{dn} = \left(\frac{\partial G}{\partial \zeta}\right)_{\zeta=0} = 2\left(\frac{z}{r^3}\right)_{\zeta=0} = -2\left(\frac{\partial \frac{1}{r}}{\partial z}\right)_{\zeta=0},$$

equations (5') and similar ones immediately give us

$$\left.\begin{aligned} u &= -\frac{1}{2\pi}\frac{\partial}{\partial z}\int_\sigma \frac{u}{r}\,d\sigma + \frac{\lambda+\mu}{4\pi\mu}\,z\,\frac{\partial}{\partial x}\int_\sigma \frac{\theta}{r}\,d\sigma, \\ v &= -\frac{1}{2\pi}\frac{\partial}{\partial z}\int_\sigma \frac{v}{r}\,d\sigma + \frac{\lambda+\mu}{4\pi\mu}\,z\,\frac{\partial}{\partial y}\int_\sigma \frac{\theta}{r}\,d\sigma, \\ w &= -\frac{1}{2\pi}\frac{\partial}{\partial z}\int_\sigma \frac{w}{r}\,d\sigma + \frac{\lambda+\mu}{4\pi\mu}\,z\,\frac{\partial}{\partial z}\int_\sigma \frac{\theta}{r}\,d\sigma, \end{aligned}\right\} \tag{10}$$

while θ will be given by

$$\theta = -\frac{\mu}{\pi(\lambda+3\mu)}\frac{\partial}{\partial z}\left[\frac{\partial}{\partial x}\int_\sigma \frac{u}{r}\,d\sigma + \frac{\partial}{\partial y}\int_\sigma \frac{v}{r}\,d\sigma + \frac{\partial}{\partial z}\int_\sigma \frac{w}{r}\,d\sigma\right], \tag{11}$$

by which

$$\int_\sigma \frac{\theta}{r}\,d\sigma = \frac{2\mu}{\lambda+3\mu}\left[\frac{\partial}{\partial x}\int_\sigma \frac{u}{r}\,d\sigma + \frac{\partial}{\partial y}\int_\sigma \frac{v}{r}\,d\sigma + \frac{\partial}{\partial z}\int_\sigma \frac{w}{r}\,d\sigma\right]. \tag{12}$$

Contrariwise, assuming that functions u, v, w given on σ are finite amd continuous functions of the points in plane σ which also have finite partial derivatives of the first order with respect to x and y, and that in the points at infinity of the same plane σ they become zero with orders superior to that of $\frac{1}{r}$, the integrals

$$\int_\sigma \frac{u}{r}\,d\sigma, \quad \int_\sigma \frac{v}{r}\,d\sigma, \quad \int_\sigma \frac{w}{r}\,d\sigma$$

are finite and continuous functions in the whole space. Their first derivatives /137

$$\frac{\partial}{\partial x}\int_\sigma \frac{u}{r}\,d\sigma = \int_\sigma \frac{\partial u}{\partial \xi}\frac{d\sigma}{r}, \quad \frac{\partial}{\partial y}\int_\sigma \frac{u}{r}\,d\sigma = \int_\sigma \frac{\partial u}{\partial \eta}\frac{\partial\sigma}{r}, \quad \frac{\partial}{\partial z}\int_\sigma \frac{u}{r}\,d\sigma; \dots$$

and their second derivatives

$$\frac{\partial^2}{\partial x^2}\int_\sigma \frac{u}{r}\,d\sigma = \frac{\partial}{\partial x}\int_\sigma \frac{\partial u}{\partial \xi}\frac{d\sigma}{r}, \quad \frac{\partial^2}{\partial y^2}\int_\sigma \frac{u}{r}\,d\sigma = \frac{\partial}{\partial y}\int_\sigma \frac{\partial u}{\partial \eta}\frac{\partial\sigma}{r}, \quad \frac{\partial^2}{\partial z^2}\int_\sigma \frac{u}{r}\,d\sigma =$$

$$= -\frac{\partial^2}{\partial x^2}\int_\sigma \frac{u}{r}\,d\sigma - \frac{\partial^2}{\partial y^2}\int_\sigma \frac{u}{r}\,d\sigma; \dots$$

are finite and continuous functions in S and tend toward finite limits when we approach the points in σ; functions u, v, w given by equation (10), when it is assumed that integral $\int_\sigma \frac{\theta}{r} d\sigma$ is given by equation (12), are regular and identically satisfy equations (1) in S, whatever may be the values of the constants λ and μ. Moreover, when we approach the points in σ they tend toward the corresponding values assigned to these points. Function θ, finally, given by equation (11) is harmonic and regular in S and on σ tends toward finite values.

2. *Case in which L, M, N are given on the limiting plane.* Green's function G_1 relative to space S and to point (x, y, z) reduces to

$$\frac{1}{r}+\frac{1}{r_1}$$

and therefore on σ takes on the value of $\frac{2}{r}$. If for $\zeta = 0$ it is noted that

$$\cos n\xi = 0,\ \cos n\eta = 0,\ \cos n\zeta = 1,\ \frac{d}{dn} = \frac{\partial}{\partial \zeta},$$

equations (8) immediately give us

$$\left.\begin{aligned}
u &= \frac{1}{4\pi\mu}\int_\sigma \frac{L}{r}\, d\sigma - \frac{1}{2\pi}\int_\sigma \frac{\varpi_2}{r}\, d\sigma + \frac{\lambda+\mu}{4\pi\mu}\frac{\partial}{\partial x}\int_\sigma \frac{\partial\theta}{\partial\zeta} r\, d\sigma,\\
v &= \frac{1}{4\pi\mu}\int_\sigma \frac{M}{r}\, d\sigma + \frac{1}{2\pi}\int_\sigma \frac{\varpi_1}{r}\, d\sigma + \frac{\lambda+\mu}{4\pi\mu}\frac{\partial}{\partial y}\int_\sigma \frac{\partial\theta}{\partial\zeta} r\, d\sigma,\\
w &= \frac{1}{4\pi\mu}\int_\sigma \frac{N}{r}\, d\sigma - \frac{1}{4\pi}\int_\sigma \frac{\theta}{r}\, d\sigma + \frac{\lambda+\mu}{4\pi\mu}\frac{\partial}{\partial z}\int_\sigma \frac{\partial\theta}{\partial\zeta} r\, d\sigma
\end{aligned}\right\} \tag{13}$$

and to complete the solution of our problem the harmonic functions θ, $\bar{\omega}_1$, $\bar{\omega}_2$ /138
of the following equations

$$\left.\begin{aligned}
\frac{2\lambda+3\mu}{2\mu}\theta &= \frac{1}{4\pi\mu}\left[\frac{\partial}{\partial x}\int_\sigma \frac{L}{r}\, d\sigma + \frac{\partial}{\partial y}\int_\sigma \frac{M}{r}\, d\sigma + \frac{\partial}{\partial z}\int_\sigma \frac{N}{r}\, d\sigma\right] -\\
&\qquad - \frac{1}{2\pi}\left[\frac{\partial}{\partial x}\int_\sigma \frac{\varpi_2}{r}\, d\sigma - \frac{\partial}{\partial y}\int_\sigma \frac{\varpi_1}{n}\, d\sigma\right],\\
\varpi_1 &= \frac{1}{4\pi\mu}\left[\frac{\partial}{\partial y}\int_\sigma \frac{N}{r}\, d\sigma - \frac{\partial}{\partial z}\int_\sigma \frac{M}{r}\, d\sigma\right] - \frac{1}{4\pi}\frac{\partial}{\partial y}\int_\sigma \frac{\theta}{r}\, d\sigma,\\
\varpi_2 &= \frac{1}{4\pi\mu}\left[\frac{\partial}{\partial z}\int_\sigma \frac{L}{r}\, d\sigma - \frac{\partial}{\partial x}\int_\sigma \frac{N}{r}\, d\sigma\right] + \frac{1}{4\pi}\frac{\partial}{\partial x}\int_\sigma \frac{\theta}{r}\, d\sigma.
\end{aligned}\right\} \tag{14}$$

remain to be determined.

To find θ, $\overline{\omega}_1$, $\overline{\omega}_2$, from these equations, let us note that since

$$\frac{\partial}{\partial x}\int_\sigma \frac{\theta}{r}\,d\sigma = \int_\sigma \frac{\partial\theta}{\partial\xi}\frac{d\sigma}{r},\quad \frac{\partial}{\partial y}\int_\sigma \frac{\theta}{r}\,d\sigma = \int_\sigma \frac{\partial\theta}{\partial\eta}\frac{\partial\sigma}{r},\quad \frac{\partial}{\partial z}\int_\sigma \frac{\theta}{r}\,d\sigma = \int_\sigma \frac{\partial\theta}{\partial\zeta}\frac{\partial\sigma}{r};\ldots,$$

the same differential relationships hold between integrals

$$\int_\sigma \frac{\theta}{r}\,d\sigma;\ \int_\sigma \frac{\varpi_1}{r}\,d\sigma,\ \int_\sigma \frac{\varpi_2}{r}\,d\sigma,\ \frac{\varpi_3}{r}\,d\sigma,$$

as between the quantities θ; $\overline{\omega}_1$, $\overline{\omega}_2$, $\overline{\omega}_3$. Hence

$$\frac{\partial}{\partial x}\int_\sigma \frac{\varpi_2}{r}\,d\sigma - \frac{\partial}{\partial y}\int_\sigma \frac{\varpi_1}{r}\,d\sigma = \frac{\lambda+2\mu}{2\mu}\frac{\partial}{\partial z}\int_\sigma \frac{\theta}{r}\,d\sigma = -\frac{\lambda+2\mu}{\mu}\pi\theta$$

and the first equation of expression (14) gives us

$$\theta = \frac{1}{2\pi(\lambda+\mu)}\left[\frac{\partial}{\partial x}\int_\sigma \frac{L}{r}\,d\sigma + \frac{\partial}{\partial y}\int_\sigma \frac{M}{r}\,d\sigma + \frac{\partial}{\partial z}\int_\sigma \frac{N}{r}\,d\sigma\right] \tag{15}$$

from which we have

$$\left.\begin{aligned}\int_\sigma \frac{\theta}{r}\,d\sigma = -\frac{1}{\lambda+\mu}\Big[\frac{\partial}{\partial x}\int_\sigma L\log(z+r)\,d\sigma &+ \\ +\frac{\partial}{\partial y}\int_\sigma M\log(z+r)\,d\sigma + \frac{\partial}{\partial z}\int_\sigma N\log(z+r)\,d\sigma\Big]&.\end{aligned}\right\} \tag{16}$$

The other two equations of expression (14) then immediately give us /139

$$\left.\begin{aligned}\varpi_1 &= \frac{1}{4\pi\mu}\left[\frac{\partial}{\partial y}\int_\sigma \frac{N}{r}\,d\sigma - \frac{\partial}{\partial z}\int_\sigma \frac{M}{r}\,d\sigma\right] + \\ &+ \frac{1}{4\pi(\lambda+\mu)}\frac{\partial}{\partial y}\Big[\frac{\partial}{\partial x}\int_\sigma L\log(z+r)\,d\sigma + \\ &+ \frac{\partial}{\partial y}\int_\sigma M\log(z+r)\,d\sigma + \frac{\partial}{\partial z}\int_\sigma N\log(z+r)\,d\sigma\Big], \\ \varpi_2 &= \frac{1}{4\pi\mu}\left[\frac{\partial}{\partial z}\int_\sigma \frac{L}{r}\,d\sigma - \frac{\partial}{\partial x}\int_\sigma \frac{N}{r}\,d\sigma\right] - \\ &- \frac{1}{4\pi(\lambda+\mu)}\frac{\partial}{\partial x}\Big[\frac{\partial}{\partial x}\int_\sigma L\log(z+r)\,d\sigma + \\ &+ \frac{\partial}{\partial y}\int_\sigma M\log(z+r)\,d\sigma + \frac{\partial}{\partial z}\int_\sigma N\log(z+r)\,d\sigma\Big].\end{aligned}\right\} \tag{17}$$

and all the elements of the problem have been determined.

Inversely, if we assume that L, M, N are finite and continuous functions of the points in σ and become zero at infinity with an order higher than $\frac{1}{r^2}$, the integrals

$$\int_\sigma \frac{L}{r}\, d\sigma, \ldots \quad \int_\sigma L \log (z + r)\, d\sigma; \ldots$$

will be finite and continuous harmonic functions in all of S. The first-order derivatives of the first type and the first- and second-order derivatives of integrals of the second type will converge to finite limits in the points in σ, for which the functions θ, $\overline{\omega}_1$, $\overline{\omega}_2$ determined from expressions (15) and (17) will be harmonic and regular in S and will converge toward finite limits on σ. If then we note that

$$\int_\sigma \frac{\partial \theta}{\partial \zeta} r\, d\sigma = z \int_\sigma \frac{\theta}{r}\, d\sigma - \int_\sigma \theta \log (z + r)\, d\sigma,$$

it turns out that also the values of u, v, w determined from expression (13) are regular functions in S and identically satisfy expression (1), whatever λ and μ may be. If, finally, we note that for expression (13) /140

$$\frac{\partial u}{\partial z} = \frac{1}{4\pi\mu} \frac{\partial}{\partial z} \int_\sigma \frac{L}{r}\, d\sigma + \varpi_2 - \frac{\lambda + \mu}{2\mu} z \frac{\partial \theta}{\partial x},$$

$$\frac{\partial v}{\partial z} = \frac{1}{4\pi\mu} \frac{\partial}{\partial z} \int_\sigma \frac{M}{r}\, d\sigma - \varpi_1 - \frac{\lambda + \mu}{2\mu} z \frac{\partial \theta}{\partial z},$$

$$\frac{\partial w}{\partial z} = \frac{1}{4\pi\mu} \frac{\partial}{\partial z} \int_\sigma \frac{N}{r}\, d\sigma - \frac{\lambda}{2\mu} \theta - \frac{\lambda + \mu}{2\mu} z \frac{\partial \theta}{\partial z},$$

under the assumption that derivatives $\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}$ have finite limits on σ,

which is certainly obtained if it is assumed that L, M, N have their finite first derivatives on σ, it is found that

$$\lim_{z=0} 2\mu \left(\frac{\partial u}{\partial z} - \varpi_2\right) = \lim_{z=0} \frac{1}{2\pi} \frac{\partial}{\partial z} \int_\sigma \frac{L}{r}\, d\sigma = -L,$$

$$\lim_{z=0} 2\mu \left(\frac{\partial v}{\partial z} + \varpi_1\right) = -M, \quad \lim_{z=0} \left(2\mu \frac{\partial w}{\partial z} + \lambda \theta\right) = -N,$$

and the neighboring conditions are also verified.

By making the formulation

$$\mathfrak{L} = \int_\sigma L \log (z + r)\, d\sigma, \quad \mathfrak{M} = \int_\sigma M \log (z + r)\, d\sigma, \quad \mathfrak{N} = \int_\sigma N \log (z + r)\, d\sigma,$$

$$\overline{\mathfrak{L}}=\int_\sigma L\Big(z\log(z+r)-r\Big)d\sigma,\quad \overline{\mathfrak{M}}=\int_\sigma M\Big(z\log(z+r)-r\Big)d\sigma,$$

$$\overline{\mathfrak{N}}=\int_\sigma N\Big(z\log(z+r)-r\Big)d\sigma,$$

$$\psi=\frac{\partial\mathfrak{L}}{\partial x}+\frac{\partial\mathfrak{M}}{\partial y}+\frac{\partial\mathfrak{N}}{\partial z},\quad \chi=\frac{\partial\overline{\mathfrak{L}}}{\partial x}+\frac{\partial\overline{\mathfrak{M}}}{\partial y}+\frac{\partial\overline{\mathfrak{N}}}{\partial z},$$

it is easily found that

$$\int_\sigma\frac{\theta}{r}d\sigma=-\frac{\psi}{\lambda+\mu},\quad \int_\sigma\theta\log(z+r)\,d\sigma=-\frac{\chi}{\lambda+\mu},$$

$$\int_\sigma\frac{\varpi_1}{r}d\sigma=-\frac{1}{2\mu}\left(\frac{\partial\mathfrak{N}}{\partial y}-\frac{\partial\mathfrak{M}}{\partial z}\right)-\frac{1}{2(\lambda+\mu)}\frac{\partial\chi}{\partial y},$$

$$\int_\sigma\frac{\varpi_2}{r}d\sigma=-\frac{1}{2\mu}\left(\frac{\partial\mathfrak{L}}{\partial z}-\frac{\partial\mathfrak{N}}{\partial x}\right)+\frac{1}{2(\lambda+\mu)}\frac{\partial\chi}{\partial x},$$

and expressions (13) take the form which was given them by Professor Cerruti: /141

$$\left.\begin{aligned}u&=\frac{1}{4\pi\mu}\frac{\partial\mathfrak{L}}{\partial z}-\frac{1}{4\pi(\lambda+\mu)}\frac{\partial\chi}{\partial x}-\frac{z}{4\pi\mu}\frac{\partial\psi}{\partial x}+\frac{1}{4\pi\mu}\frac{\partial}{\partial y}\left(\frac{\partial\overline{\mathfrak{M}}}{\partial x}-\frac{\partial\overline{\mathfrak{L}}}{\partial y}\right),\\ v&=\frac{1}{4\pi\mu}\frac{\partial\mathfrak{M}}{\partial z}-\frac{1}{4\pi(\lambda+\mu)}\frac{\partial\chi}{\partial y}-\frac{z}{4\pi\mu}\frac{\partial\psi}{\partial y}-\frac{1}{4\pi\mu}\frac{\partial}{\partial x}\left(\frac{\partial\overline{\mathfrak{M}}}{\partial x}-\frac{\partial\overline{\mathfrak{L}}}{\partial y}\right),\\ w&=\frac{1}{4\pi\mu}\frac{\partial\mathfrak{N}}{\partial z}+\frac{\psi}{4\pi(\lambda+\mu)}-\frac{z}{4\pi\mu}\frac{\partial\psi}{\partial z}.\end{aligned}\right\}\tag{13}$$

3. *Case in which u, v, N are given on the limiting plane.* To solve the problem now posed, we will make use of the formulas

$$\left.\begin{aligned}u&=-\frac{1}{2\pi}\frac{\partial}{\partial z}\int_\sigma\frac{u}{r}d\sigma+\frac{\lambda+\mu}{4\pi\mu}z\frac{\partial}{\partial x}\int_\sigma\frac{\theta}{r}d\sigma,\\ v&=-\frac{1}{2\pi}\frac{\partial}{\partial z}\int_\sigma\frac{v}{r}d\sigma+\frac{\lambda+\mu}{4\pi\mu}z\frac{\partial}{\partial y}\int_\sigma\frac{\theta}{r}d\sigma,\\ w&=\frac{1}{4\pi\mu}\int_\sigma\frac{N}{r}d\sigma-\frac{1}{4\pi}\int_\sigma\frac{\theta}{r}d\sigma+\frac{\lambda+\mu}{4\pi\mu}z\frac{\partial}{\partial z}\int_\sigma\frac{\theta}{r}d\sigma.\end{aligned}\right\}\tag{18}$$

The problem is solved as soon as we succeed in finding θ, and this function is immediately given by equation

$$\theta=-\frac{\mu}{\pi(\lambda+2\mu)}\frac{\partial}{\partial z}\left[\frac{\partial}{\partial x}\int_\sigma\frac{u}{r}ds+\frac{\partial}{\partial y}\int_\sigma\frac{v}{r}d\sigma-\frac{1}{2\mu}\int_\sigma\frac{N}{r}d\sigma\right].\tag{19}$$

Under conditions which are easy to investigate, it is conversely shown that for functions u, v, N on plane σ, all the conditions of the problem are satisfied.

4. *Case in which L, M, w are given on the limiting plane.* This new problem is also easily solved with the formulas

$$
\left.\begin{aligned}
u &= \frac{1}{4\pi\mu}\int\limits_{\sigma}\frac{L}{r}\,d\sigma - \frac{1}{2\pi}\int\limits_{\sigma}\frac{\varpi_2}{r}\,d\sigma + \frac{\lambda+\mu}{4\pi\mu}\frac{\partial}{\partial x}\int\limits_{\sigma}\frac{\partial\theta}{\partial\zeta}r\,d\sigma,\\
v &= \frac{1}{4\pi\mu}\int\limits_{\sigma}\frac{M}{r}\,d\sigma + \frac{1}{2\pi}\int\limits_{\sigma}\frac{\varpi_1}{r}\,d\sigma + \frac{\lambda+\mu}{4\pi\mu}\frac{\partial}{\partial y}\int\limits_{\sigma}\frac{\partial\theta}{\partial\zeta}r\,d\sigma,\\
w &= -\frac{1}{2\pi}\frac{\partial}{\partial z}\int\limits_{\sigma}\frac{w}{r}\,d\sigma \qquad\qquad\quad + \frac{\lambda+\mu}{4\pi\mu}\frac{\partial}{\partial z}\int\limits_{\sigma}\frac{\partial\theta}{\partial\zeta}r\,d\sigma,
\end{aligned}\right\} \tag{20}
$$

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and by determining θ, $\overline{\omega}_1$, $\overline{\omega}_2$ from equations

$$
\left.\begin{aligned}
\frac{\lambda+2\mu}{\mu}\theta &= \frac{1}{2\pi}\left[\frac{1}{2\mu}\frac{\partial}{\partial x}\int\limits_{\sigma}\frac{L}{r}\,d\sigma + \frac{1}{2\mu}\frac{\partial}{\partial y}\int\limits_{\sigma}\frac{M}{r}\,d\sigma - \frac{\partial^2}{\partial z^2}\int\limits_{\sigma}\frac{w}{r}\,d\sigma\right] -\\
&\qquad - \frac{1}{2\pi}\left[\frac{\partial}{\partial x}\int\limits_{\sigma}\frac{\varpi_2}{r}\,d\sigma - \frac{\partial}{\partial y}\int\limits_{\sigma}\frac{\varpi_1}{r}\,d\sigma\right],\\
\varpi_1 &= -\frac{1}{2\pi}\frac{\partial}{\partial z}\left[\frac{\partial}{\partial y}\int\limits_{\sigma}\frac{w}{r}\,d\sigma + \frac{1}{2\mu}\int\limits_{\sigma}\frac{M}{r}\,d\sigma\right],\\
\varpi_2 &= \frac{1}{2\pi}\frac{\partial}{\partial z}\left[\frac{1}{2\mu}\int\limits_{\sigma}\frac{L}{r}\,d\sigma + \frac{\partial}{\partial x}\int\limits_{\sigma}\frac{w}{r}\,d\sigma\right].
\end{aligned}\right\} \tag{21}
$$

These formulas solve the problem. This can be established directly when it is assumed that the knowns satisfy conditions on the plane which are similar to those which we have established in the preceding cases.

We may also add that from the last two equations in expression (21) we have

$$
-\frac{1}{2\pi}\left[\frac{\partial}{\partial x}\int\limits_{\sigma}\frac{\varpi_2}{r}\,d\sigma - \frac{\partial}{\partial y}\int\limits_{\sigma}\frac{\varpi_1}{r}\,d\sigma\right] = \frac{1}{2\pi}\left[\frac{1}{2\mu}\frac{\partial}{\partial x}\int\limits_{\sigma}\frac{L}{r}\,d\sigma + \frac{1}{2\mu}\frac{\partial}{\partial y}\int\limits_{\sigma}\frac{M}{r}\,d\sigma - \frac{\partial^2}{\partial z^2}\int\limits_{\sigma}\frac{w}{r}\,d\sigma\right]
$$

and that, therefore, θ is given simply by formula

$$
\theta = \frac{\mu}{\pi(\lambda+2\mu)}\left[\frac{1}{2\mu}\frac{\partial}{\partial x}\int\limits_{\sigma}\frac{L}{r}\,d\sigma + \frac{1}{2\mu}\frac{\partial}{\partial y}\int\limits_{\sigma}\frac{M}{r}\,d\sigma - \frac{\partial^2}{\partial z^2}\int\limits_{\sigma}\frac{w}{r}\,d\sigma\right]. \tag{21'}
$$

5. *Cases in which u, M, w or L, v, w are given on the limiting plane.* These two new problems are identical, since one is derived from the other by interchanging the x and y-axes. We will assume that u, M, w are given, and will write the formulas

$$
u = -\frac{1}{2\pi}\frac{\partial}{\partial z}\int\limits_{\sigma}\frac{u}{r}\,d\sigma + \frac{\lambda+\mu}{4\pi\mu}z\frac{\partial}{\partial x}\int\limits_{\sigma}\frac{\theta}{r}\,d\sigma, \tag{22}
$$

$$
\left.\begin{aligned}
v &= \frac{1}{4\pi\mu}\int_\sigma \frac{M}{r}\,d\sigma + \frac{1}{2\pi}\int_\sigma \frac{\varpi_1}{r}\,d\sigma + \\
&\quad + \frac{\lambda+\mu}{4\pi\mu}\left(z\frac{\partial}{\partial y}\int_\sigma \frac{\theta}{r}\,d\sigma - \frac{\partial}{\partial y}\int_\sigma \theta\log(z+r)\,d\sigma\right), \\
w &= -\frac{1}{2\pi}\frac{\partial}{\partial z}\int_\sigma \frac{w}{r}\,d\sigma + \frac{\lambda+\mu}{4\pi\mu}\,z\frac{\partial}{\partial z}\int_\sigma \frac{\theta}{r}\,d\sigma.
\end{aligned}\right\} \tag{22}
$$

In these formulas the only unknowns appearing are θ and $\bar{\omega}_1$. Therefore, to solve the problem it is sufficent to determine these two magnitudes. The second of them is immediately given by formula /143

$$
\varpi_1 = -\frac{1}{2\pi}\frac{\partial}{\partial z}\left[\frac{\partial}{\partial y}\int_\sigma \frac{w}{r}\,d\sigma + \frac{1}{2\mu}\int_\sigma \frac{M}{r}\,d\sigma\right], \tag{23}
$$

while θ must be derived from an equation which, by taking advantage of equation (23), is reduced to

$$
\left.\begin{aligned}
\frac{\lambda+3\mu}{2\mu}\,\theta &= \frac{1}{2\pi}\frac{\partial}{\partial z}\left[-\frac{\partial}{\partial x}\int_\sigma \frac{u}{r}\,d\sigma + \frac{1}{\mu}\frac{\partial}{\partial y}\int_\sigma M\log(z+r)\,d\sigma - \right. \\
&\quad \left. - \frac{\partial}{\partial z}\int_\sigma \frac{w}{r}\,d\sigma + \frac{\partial^2}{\partial y^2}\int_\sigma w\log(z+r)\,d\sigma\right] - \\
&\quad - \frac{\lambda+\mu}{4\pi\mu}\frac{\partial^2}{\partial y^2}\int_\sigma \theta\log(z+r)\,d\sigma.
\end{aligned}\right\} \tag{24}
$$

Now since θ is harmonic and moreover equals

$$
-\frac{1}{2\pi}\frac{\partial^2}{\partial z^2}\int_\sigma \theta\log(z+r)\,d\sigma,
$$

it is easily found by setting

$$
\varphi = \int_\sigma \theta\log(z+r)\,d\sigma,
$$

that equation (24) may be given the form

$$
\left.\begin{aligned}
\frac{\lambda+3\mu}{2\mu}\frac{\partial^2\varphi}{\partial x^2} + \frac{\lambda+2\mu}{\mu}\frac{\partial^2\varphi}{\partial y^2} &= \frac{\partial}{\partial z}\left[-\frac{\partial}{\partial x}\int_\sigma \frac{u}{r}\,d\sigma + \right. \\
+ \frac{1}{\mu}\frac{\partial}{\partial y}\int_\sigma M\log(z+r)\,d\sigma - \frac{\partial}{\partial z}\int_\sigma \frac{w}{r}\,d\sigma &+ \left.\frac{\partial^2}{\partial y^2}\int_\sigma w\log(z+r)\,d\sigma\right].
\end{aligned}\right\} \tag{24'}
$$

We can easily demonstrate the fact that there can be but a single function θ, harmonic and regular in S, such that the corresponding function ϕ satisfies expression (24'). If, in fact, two functions -- θ and θ' -- could be determined, by calling ϕ' the function analagous to ϕ and related to θ', the difference

$$
\psi = \varphi - \varphi'
$$

would have to satisfy equation /144

$$\frac{\lambda+3\mu}{2\mu}\frac{\partial^2\psi}{\partial x^2}+\frac{\lambda+2\mu}{\mu}\frac{\partial^2\psi}{\partial y^2}=0.$$

Now by setting

$$t_1=x\sqrt{\frac{\lambda+2\mu}{\mu}}+iy\sqrt{\frac{\lambda+3\mu}{2\mu}},\quad t_2=x\sqrt{\frac{\lambda+2\mu}{\mu}}-iy\sqrt{\frac{\lambda+3\mu}{2\mu}},$$

the general integral of the last equation is given by

$$\psi=f_1(t_1)+f_2(t_2)+f_3(z)$$

where f_1, f_2, f_3 are arbitrary functions. But since t_1, t_2, and z also may be regarded as arbitrary parameters in order that ψ satisfy equation $\Delta^2 = 0$, it must be the case that

$$f''_1=\text{cost.},\quad f''_2=\text{cost.},\qquad f''_3=\text{cost.}$$

If it is then noted that θ must also become zero at infinity, it is found that $f''_3 = 0$ and therefore in identical fashion

$$\theta-\theta'=-\frac{1}{2\pi}\frac{\partial^2\psi}{\partial z^2}=0.$$

On the other hand, if we call ξ', η' the coordinates of x and y which appear on the right side of equation (24'), it is easily found that the function of x, y, z satisfies

$$\left.\begin{aligned}\varphi=\frac{1}{2\pi}\frac{\partial}{\partial z}\int_\sigma d\sigma'\Big[-\frac{\partial}{\partial\xi'}\int_\sigma\frac{u}{r}\,d\sigma+\\+\frac{1}{\mu}\frac{\partial}{\partial\eta'}\int_\sigma M\log(z+r)\,d\sigma-\frac{\partial}{\partial z}\int_\sigma\frac{w}{r}\,d\sigma+\\+\frac{\partial^2}{\partial\eta'^2}\int_\sigma w\log(z+r)\,d\sigma\Big]\log\sqrt{\frac{2\mu}{\lambda+3\mu}(x-\xi')^2+\frac{\mu}{\lambda+2\mu}(y-\eta')^2}\end{aligned}\right\}\qquad(25)$$

in which $d\sigma' = d\xi'd\eta'$ satisfies equation (24') and is harmonic. The first property is evidently fulfilled since the function of x, y

$$\log\sqrt{\frac{2\mu}{\lambda+3\mu}(x-\xi')^2+\frac{\mu}{\lambda+2\mu}(y-\eta')^2}$$

satisfies equation (24') when the right side is zero, and becomes infinite when $x = \xi'$, $y = \eta'$ in the manner of log r. As for the second property, it should be noted that, if χ is a function of the points in plane σ which becomes zero at infinity, together with the first derivatives /145

$$\frac{\partial}{\partial x}\int_\sigma\chi\log\sqrt{\frac{2\mu}{\lambda+3\mu}(x-\xi')^2+\frac{\mu}{\lambda+2\mu}(y-\eta')^2}\,d\sigma'=$$

$$=\int_\sigma\frac{\partial\chi}{\partial\xi}\log\sqrt{\frac{2\mu}{\lambda+3\mu}(x-\xi')^2+\frac{\mu}{\lambda+2\mu}(y-\eta')^2}\,d\sigma',\ldots$$

and therefore

$$\Delta^2 \int_\sigma \chi \log \sqrt{\frac{2\mu}{\lambda+3\mu}(x-\xi')^2 + \frac{\mu}{\lambda+2\mu}(\eta-\eta')^2}\, d\sigma' =$$
$$= \int_\sigma \Delta^2 \chi \log \sqrt{\frac{2\mu}{\lambda+3\mu}(x-\xi)^2 + \frac{\mu}{\lambda+2\mu}(\eta-\eta')^2}\, d\sigma',$$

for which ϕ is a harmonic function which becomes zero at infinity.

From expression (25) it is immediately found that

$$\left.\begin{aligned}
\int_\sigma \frac{\theta}{r} d\sigma = \frac{1}{2\pi}\frac{\partial^2}{\partial z^2}\int_\sigma d\sigma' \Big[-\frac{\partial}{\partial \xi'}\int_\sigma \frac{u}{r} d\sigma + \\
+ \frac{1}{\mu}\frac{\partial}{\partial \eta'}\int_\sigma M \log(z+r)\, d\sigma - \frac{\partial}{\partial z}\int_\sigma \frac{w}{r} d\sigma + \\
+ \frac{\partial^2}{\partial \eta'^2}\int_\sigma w \log(z+r)\, d\sigma \Big] \log\sqrt{\frac{2\mu}{\lambda+3\mu}(x-\xi')^2 + \frac{\mu}{\lambda+2\mu}(y-\eta')^2}, \\
\theta = -\frac{1}{2^2\pi^2}\frac{\partial^3}{\partial z^3}\int_\sigma d\sigma' \Big[-\frac{\partial}{\partial \xi'}\int_\sigma \frac{u}{r} d\sigma + \\
+ \frac{1}{\mu}\frac{\partial}{\partial \eta'}\int_\sigma M \log(z+r)\, d\sigma - \frac{\partial}{\partial z}\int_\sigma \frac{w}{r} d\sigma + \\
+ \frac{\partial^2}{\partial \eta'^2}\int_\sigma w \log(z+r)\, d\sigma \Big] \log\sqrt{\frac{2\mu}{\lambda+3\mu}(x-\xi')^2 + \frac{\mu}{\lambda+2\mu}(y-\eta')^2}.
\end{aligned}\right\} \qquad (26)$$

Conversely, if u, M, w on σ are given as definite and continuous functions having finite partial derivatives of the first order, and moreover u, w become zero in the points at infinity in σ with order higher than $\frac{1}{r}$, while M becomes zero with an order higher than $\frac{1}{r^2}$, the right side of equation (24) will be a harmonic function regular in S, becoming zero at infinity, and tending toward finite values on σ. Under these conditions, functions ϕ, $\int_\sigma \frac{\theta}{r} d\sigma$ and θ, given by expressions (25) and (26), are harmonic and regular in S and tend toward finite values when we approach the points on σ. The values of u, v, w given by expression (22) are regular in S, identically satisfy expression (1) in this same space, and satisfy the boundary conditions, as is easily demonstrated. /146

6. *Cases in which u, M, N or L, v, N are given on the limiting plane.* These two problems also differ only by the interchange of the x and y-axes and are solved like the foregoing. Let us consider the case in which u, M, N are given, and we will write the equations

$$u = -\frac{1}{2\pi}\frac{\partial}{\partial z}\int_\sigma \frac{u}{r} d\sigma + \frac{\lambda+\mu}{4\pi\mu} z \frac{\partial}{\partial x}\int_\sigma \frac{\theta}{r} d\sigma, \qquad (27)$$

$$\left.\begin{aligned} v &= \frac{1}{4\pi\mu}\int_\sigma \frac{M}{r}\,d\sigma + \frac{1}{2\pi}\int_\sigma \frac{\varpi_1}{r}\,d\sigma + \frac{\lambda+\mu}{4\pi\mu}\left(z\frac{\partial}{\partial y}\int_\sigma \frac{\theta}{r}\,d\sigma - \right. \\ &\qquad \left. - \frac{\partial}{\partial y}\int_\sigma \theta\log(z+r)\,d\sigma\right), \\ w &= \frac{1}{4\pi\mu}\int_\sigma \frac{N}{r}\,d\sigma - \frac{1}{4\pi}\int_\sigma \frac{\theta}{r}\,d\sigma + \frac{\lambda+\mu}{4\pi\mu} z\frac{\partial}{\partial z}\int_\sigma \frac{\theta}{r}\,d\sigma, \end{aligned}\right\} \tag{27}$$

It will be sufficient to find functions θ and $\bar{\omega}_1$ from the two equations

$$\begin{aligned} \frac{\lambda+2\mu}{2\mu}\theta &= \frac{1}{2\pi}\left[-\frac{\partial^2}{\partial x\,\partial z}\int_\sigma \frac{u}{r}\,d\sigma + \frac{1}{2\mu}\frac{\partial}{\partial y}\int_\sigma \frac{M}{r}\,d\sigma + \frac{1}{2\mu}\frac{\partial}{\partial z}\int_\sigma \frac{N}{r}\,d\sigma\right] + \\ &\qquad + \frac{1}{2\pi}\frac{\partial}{\partial y}\int_\sigma \frac{\varpi_1}{r}\,d\sigma - \frac{\lambda+\mu}{4\pi\mu}\frac{\partial^2}{\partial y^2}\int_\sigma \theta\log(z+r)\,d\sigma, \\ \varpi_1 &= \frac{1}{4\pi\mu}\left[\frac{\partial}{\partial y}\int_\sigma \frac{N}{r}\,d\sigma - \frac{\partial}{\partial z}\int_\sigma \frac{M}{r}\,d\sigma\right] - \frac{1}{4\pi}\frac{\partial}{\partial y}\int_\sigma \frac{\theta}{r}\,d\sigma. \end{aligned} \tag{28}$$

Now from the second of these equations we derive /147

$$\begin{aligned} \int_\sigma \frac{\varpi_1}{r}\,d\sigma &= -\frac{1}{2\mu}\left[\frac{\partial}{\partial y}\int_\sigma N\log(z+r)\,d\sigma - \int_\sigma \frac{M}{r}\,d\sigma\right] + \\ &\qquad + \frac{1}{2}\frac{\partial}{\partial y}\int_\sigma \theta\log(z+r)\,d\sigma, \end{aligned}$$

whence, by substituting into the first and introducing function ϕ from the preceding case, it is found that ϕ must satisfy equation

$$\left.\begin{aligned} \frac{\lambda+2\mu}{2\mu}\frac{\partial^2\varphi}{\partial x^2} + \frac{\lambda+\mu}{\mu}\frac{\partial^2\varphi}{\partial y^2} &= -\frac{\partial^2}{\partial x\,\partial z}\int_\sigma \frac{u}{r}\,d\sigma + \frac{1}{\mu}\frac{\partial}{\partial y}\int_\sigma \frac{M}{r}\,d\sigma + \\ &\quad + \frac{1}{2\mu}\frac{\partial}{\partial z}\int_\sigma \frac{N}{r}\,d\sigma - \frac{1}{2\mu}\frac{\partial^2}{\partial y^2}\int_\sigma N\log(z+r)\,d\sigma. \end{aligned}\right\} \tag{29}$$

This equation is of the same type as equation (24') and function ϕ in this case, too, is determined as in the preceding case.

III. Problems in Which the Elastic Body is Limited by a Sphere.

1. *Case in which u, v, w are given on the limiting sphere. Internal space.*

Let us call R and radius of the sphere whose internal space S is occupied by the elastic body. Let us consider point (x_1, y_1, z_1), the reciprocal of point (x, y, z) with respect to the sphere of radius R, together with the point (x, y, z) inside S and the distances r, r_1 of these two points from any point (ξ, η, ζ) in S. Let us moreover set

$$l = \sqrt{x^2 + y^2 + z^2}, \qquad \rho = \sqrt{\xi^2 + \eta^2 + \zeta^2},$$
$$x\xi + y\eta + z\zeta = l\rho\cos\omega.$$

As is very well known, Green's function G relative to point (x, y, z) and to the sphere of radius R is given by

$$G=\frac{1}{r}-\frac{R}{l}\frac{1}{r_1},$$

while on surface σ of the sphere of radius R /148

$$\frac{dG}{dn}=-\left(\frac{\partial G}{\partial \rho}\right)_{\rho=R}=\left(\frac{R^2-l^2}{R r^3}\right)_{\rho=R}.$$

Consequently, expression (5') and similar ones will immediately give us:

$$\left.\begin{aligned} u&=\frac{R^2-l^2}{4\pi R}\int_\sigma \frac{u}{r^3}\,d\sigma+\frac{\lambda+\mu}{8\pi R\mu}(R^2-l^2)\frac{\partial}{\partial x}\int_\sigma \frac{\theta}{r}\,d\sigma,\\ v&=\frac{R^2-l^2}{4\pi R}\int_\sigma \frac{v}{r^3}\,d\sigma+\frac{\lambda+\mu}{8\pi R\mu}(R^2-l^2)\frac{\partial}{\partial y}\int_\sigma \frac{\theta}{r}\,d\sigma,\\ w&=\frac{R^2-l^2}{4\pi R}\int_\sigma \frac{w}{r^3}\,d\sigma+\frac{\lambda+\mu}{8\pi R\mu}(R^2-l^2)\frac{\partial}{\partial z}\int_\sigma \frac{\theta}{r}\,d\sigma, \end{aligned}\right\} \tag{30}$$

To solve the problem completely, we must only determine θ of the equation

$$\left.\begin{aligned} \theta=&\frac{\partial}{\partial x}\left(\frac{R^2-l^2}{4\pi R}\int_\sigma \frac{u}{r^3}\,d\sigma\right)+\frac{\partial}{\partial y}\left(\frac{R^2-l^2}{4\pi R}\int_\sigma \frac{v}{r^3}\,d\sigma\right)+\\ &+\frac{\partial}{\partial z}\left(\frac{R^2-l^2}{4\pi R}\int_\sigma \frac{w}{r^3}\,d\sigma\right)-\frac{\lambda+\mu}{4\pi R\mu}\,l\frac{\partial}{\partial l}\int_\sigma \frac{\theta}{r}\,d\sigma, \end{aligned}\right\} \tag{31}$$

where

$$l\frac{\partial}{\partial l}=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}.$$

Therefore, let us remember that

$$4\pi R\theta=(R^2-l^2)\int_\sigma \frac{\theta}{r^3}\,d\sigma=2l\frac{\partial}{\partial l}\int_\sigma \frac{\theta}{r}\,d\sigma+\int_\sigma \frac{\theta}{r}\,d\sigma. \tag{32}$$

From this it follows that θ is determined by equation (32) when we have found the harmonic function

$$\varphi=\int_\sigma \frac{\theta}{r}\,d\sigma$$

of the equation

$$\left.\begin{aligned} \frac{\lambda+3\mu}{\mu}\,l\frac{\partial\varphi}{\partial l}+\varphi=&\frac{\partial}{\partial x}\left[(R^2-l^2)\int_\sigma \frac{u}{r^3}\,d\sigma\right]+\\ &+\frac{\partial}{\partial y}\left[(R^2-l^2)\int_\sigma \frac{u}{r^3}\,d\sigma\right]+\frac{\partial}{\partial z}\left[(R^2-l^2)\int_\sigma \frac{w}{r}\,d\sigma\right]. \end{aligned}\right\} \tag{31'}$$

The general integral of this equation is given by /149

$$\varphi=\frac{\mu}{\lambda+3\mu}\,l^{-\frac{\mu}{\lambda+3\mu}}\int_0^l l^{-\frac{\lambda+2\mu}{\lambda+3\mu}}\left\{\frac{\partial}{\partial x}\left[(R^2-l^2)\int_\sigma \frac{u}{r^3}\,d\sigma\right]+\right\} \tag{33}$$

$$+\frac{\partial}{\partial y}\left[(R^2-l^2)\int_\sigma \frac{v}{r^3}\,d\sigma\right]+\frac{\partial}{\partial z}\left[(R^2-l^2)\int_\sigma \frac{w}{r^3}\,d\sigma\right]\Big\}\,dl+l^{-\frac{\mu}{\lambda+3\mu}}\chi\Big\} \tag{33}$$

where χ is an arbitrary function of any two parameters which, together with l, determines each point in the space. Noting that ϕ must be finite in S and that this is true of the first part on the right side of equation (33), even when $l = 0$, it follows from this that $\chi = 0$ must be true if $\frac{\mu}{\lambda+3\mu} > 0$, as is precisely the case in elasticity. The expression of ϕ thus determined is finite in the whole sphere S, including the surface, and is a harmonic function in S. This second statement is proved by noting that if

$$\varphi = l^{-c}\int_0^l l^{c-1}\psi\,dl,$$

where c is any positive constant, we also have

$$\frac{\partial\varphi}{\partial x} = l^{-c-1}\int_0^l l^c\frac{\partial\psi}{\partial x}\,dl\,(*),\ldots\qquad \Delta^2\varphi = l^{-c-2}\int_0^l l^{c+1}\Delta^2\psi\,dl$$

and thus if ψ is harmonic, then ϕ is also harmonic.

It also easily results from the fact that /150

$$(R^2-l^2)\int_\sigma\frac{u\,d\sigma}{r^3} = 2l\frac{\partial}{\partial l}\int_\sigma\frac{u\,d\sigma}{r}+\int_\sigma\frac{u\,d\sigma}{r},\ldots$$

and from known properties of the potential function that, if u, v, w are finite and continuous functions with two parameters which specify the σ points together with the first derivatives, and if they have second derivatives with respect to these finite parameters, the first derivatives of ϕ and θ strive toward finite

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* If in fact we call l, α, β the polar coordinates of a point we have

$$\frac{\partial}{\partial x} = \overset{**}{\mathrm{sen}}\,\alpha\cos\beta\frac{\partial}{\partial l}+\frac{\cos\alpha\cos\beta}{l}\frac{\partial}{\partial\alpha}-\frac{\mathrm{sen}\,\beta}{l\,\mathrm{sen}\,\alpha}\frac{\partial}{\partial\beta}$$

and hence

$$\frac{\partial}{\partial x}\left(l^{-c}\int_0^l l^{c-1}\psi\,dl\right) = -c\,l^{-c-1}\,\mathrm{sen}\,\alpha\cos\beta\int_0^l l^{c-1}\psi\,dl+\frac{\mathrm{sen}\,\alpha\cos\beta}{l}\psi+$$

$$+\,l^{-c-1}\cos\alpha\cos\beta\int_0^l l^{c-1}\frac{\partial\psi}{\partial\alpha}\,dl-l^{-c-1}\frac{\mathrm{sen}\,\beta}{\mathrm{sen}\,\alpha}\int_0^l l^{c-1}\frac{\partial\psi}{\partial\beta}\,dl=$$

$$=l^{-c-1}\int_0^l l^c\left(\mathrm{sen}\,\alpha\cos\beta\frac{\partial\psi}{\partial l}+\frac{\cos\alpha\cos\beta}{l}\frac{\partial\psi}{\partial\alpha}-\frac{\mathrm{sen}\,\beta}{l\,\mathrm{sen}\,\alpha}\frac{\partial\psi}{\partial\beta}\right)dl = l^{-c-1}\int_0^l l^c\frac{\partial\psi}{\partial x}\,dl.$$

** Sen is correctly sin in English terminology.

values on σ, and expression (1) and the neighboring conditions are identically satisfied.

1a. *External space.* Let us now resolve the same problem for the case in which the elastic body, instead of occupying the space inside the sphere of radius R, occupies all the indefinite space outside of it. Green's function G even in this case is

$$G = \frac{1}{r} - \frac{R}{l}\frac{1}{r_1}.$$

It is merely to be noted that in the present case point (x, y, z) is outside sphere σ, while (x_1, y_1, z_1) is inside. For the value of the normal derivative over σ, we have instead

$$\frac{dG}{dn} = \left(\frac{\partial G}{\partial \rho}\right)_{\rho=R} = \left(\frac{l^2 - R^2}{R r^3}\right)_{\rho=R};$$

and formulas (10) are modified as follows:

$$\left.\begin{aligned}
u &= \frac{l^2 - R^2}{4\pi R}\int_\sigma \frac{u}{r^3}\, d\sigma + \frac{\lambda+\mu}{8\pi R\mu}(l^2 - R^2)\frac{\partial}{\partial x}\int_\sigma \frac{\theta}{r}\, d\sigma,\\
v &= \frac{l^2 - R^2}{4\pi R}\int_\sigma \frac{v}{r^3}\, d\sigma + \frac{\lambda+\mu}{8\pi R\mu}(l^2 - R^2)\frac{\partial}{\partial y}\int_\sigma \frac{\theta}{r}\, d\sigma,\\
w &= \frac{l^2 - R^2}{4\pi R}\int_\sigma \frac{w}{r^3}\, d\sigma + \frac{\lambda+\mu}{8\pi R\mu}(l^2 - R^2)\frac{\partial}{\partial z}\int_\sigma \frac{\theta}{r}\, d\sigma.
\end{aligned}\right\} \tag{34}$$

The equation which determines ϕ, which in the present case is

$$-4\pi R\theta = 2l\frac{\partial}{\partial l}\int_\sigma \frac{\theta}{r}\, d\sigma + \int_\sigma \frac{\theta\, d\sigma}{r},$$

is nevertheless still equation (31'), and therefore ϕ will always be given by expression (33), in which we take ∞ and l for the integration limits. The expression for ϕ must be determined so that the limit of the product of $l\phi$ and $l = \infty$ does not become infinite. Therefore, even in this case we will have /151
to set $\chi = 0$ and will thus be able to write

$$\left.\begin{aligned}
\int_\sigma \frac{\theta}{r}\, d\sigma = \frac{\mu}{\lambda+3\mu}\, l^{-\frac{\mu}{\lambda+3\mu}}\int_\infty^l l^{-\frac{\lambda+2\mu}{\lambda+3\mu}}\Big\{\frac{\partial}{\partial x}\Big[(R^2 - l^2)\int_\sigma \frac{u}{r^3}\, d\sigma\Big] +\\
+ \frac{\partial}{\partial y}\Big[(R^2 - l^2)\int_\sigma \frac{v}{r^3}\, d\sigma\Big] + \frac{\partial}{\partial z}\Big[(R^2 - l^2)\int_\sigma \frac{w}{r^3}\Big]\Big\}\, dl.
\end{aligned}\right\} \tag{35}$$

The further considerations do not differ from those in the preceding case.

2. *Case in which L, M, N are given on the limiting sphere. Internal space.* To obtain the solution to this problem, it is well to realize that the quantities L, M, N must be assumed to satisfy the conditions

$$\int_\sigma L\,d\sigma = \int_\sigma M\,d\sigma = \int_\sigma N\,d\sigma = \int_\sigma (\eta N - \zeta M)\,d\sigma = \\ = \int_\sigma (\zeta L - \xi N)\,d\sigma = \int_\sigma (\zeta M - \eta L)\,d\sigma = 0, \tag{36}$$

which are necessary so that the elastic body will be in equilibrium.

The simplest way to solve the problem seems to me to be the following.

Let us first of all observe that equation (1) give place to others

$$\left.\begin{aligned} &\Delta^2\left(l\frac{\partial u}{\partial l}\right) + \frac{\lambda+\mu}{\mu}\frac{\partial}{\partial x}\left(l\frac{\partial\theta}{\partial l}+\theta\right)=0,\\ &\Delta^2\left(l\frac{\partial v}{\partial l}\right) + \frac{\lambda+\mu}{\mu}\frac{\partial}{\partial y}\left(l\frac{\partial\theta}{\partial l}+\theta\right)=0,\\ &\Delta^2\left(l\frac{\partial w}{\partial l}\right) + \frac{\lambda+\mu}{\mu}\frac{\partial}{\partial z}\left(l\frac{\partial\theta}{\partial l}+\theta\right)=0,\\ &l\frac{\partial\theta}{\partial l}+\theta = \frac{\partial}{\partial x}\left(l\frac{\partial u}{\partial l}\right)+\frac{\partial}{\partial y}\left(l\frac{\partial v}{\partial l}\right)+\frac{\partial}{\partial z}\left(l\frac{\partial w}{\partial z}\right), \end{aligned}\right\} \tag{37}$$

which means that equations (1) are themselves transformed into them when operation $l\frac{\partial}{\partial l}$ is applied to the unknown functions. Let us note the following formulas also:

$$l\frac{\partial\varpi_1}{\partial l}+\varpi_1 = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(l\frac{\partial w}{\partial l}\right)-\frac{\partial}{\partial z}\left(l\frac{\partial v}{\partial l}\right)\right],\ldots \tag{37'}$$

For the results of the preceding number, we may write: /152

$$\left.\begin{aligned} l\frac{\partial u}{\partial l} = \frac{R^2-l^2}{4\pi}\int_\sigma \frac{\partial u}{\partial\rho}\frac{d\sigma}{r^3} + \frac{\lambda+\mu}{8\pi\mu}(R^2-l^2)\frac{\partial}{\partial x}\int_\sigma\frac{\partial\theta}{\partial\rho}\frac{d\sigma}{r} + \\ + \frac{\lambda+\mu}{8\pi R\mu}(R^2-l^2)\frac{\partial}{\partial x}\int_\sigma\frac{\theta}{r}\,d\sigma, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{38}$$

and these formulas, being

$$\int_\sigma \frac{\partial\theta}{\partial\rho}\frac{d\sigma}{r} = \int_\sigma \theta\frac{\partial\frac{1}{r}}{\partial\rho}\,d\sigma + 4\pi\theta = -\frac{1}{2R}\int_\sigma\frac{\theta}{r}\,d\sigma + 2\pi\theta, \tag{39}$$

by Green's theorem, reduce immediately to these others:

$$l\frac{\partial u}{\partial l} = \frac{R^2-l^2}{4\pi}\int_\sigma\frac{\partial u}{\partial\rho}\frac{d\sigma}{r^3} + \frac{\lambda+\mu}{4\mu}(R^2-l^2)\frac{\partial\theta}{\partial x} + \tag{38'}$$

$$+\frac{\lambda+\mu}{16\pi R\mu}(R^2-l^2)\frac{\partial}{\partial x}\int_\sigma \frac{\theta}{r}\,d\sigma, \qquad (38')$$

$$\cdots\cdots\cdots$$

The surface conditions with

$$\cos nx = -\frac{x}{R}, \quad \cos ny = -\frac{y}{R}, \quad \cos nz = -\frac{z}{R}; \quad \frac{d}{dn} = -\frac{\partial}{\partial l},$$

on the surface become in the present case

$$L = \lambda\theta\frac{x}{R} + 2\mu\left(\frac{\partial u}{\partial l} + \frac{y}{R}\varpi_3 - \frac{z}{R}\varpi_2\right), \ldots$$

whence:

$$\frac{\partial u}{\partial l} = \frac{L}{2\mu} - \frac{\lambda}{2\mu}\frac{x}{R}\theta - \frac{y}{R}\varpi_3 + \frac{z}{R}\varpi_2, \ldots$$

for which, by eliminating $\frac{\partial u}{\partial \rho}, \frac{\partial v}{\partial \rho}, \frac{\partial w}{\partial \rho}$, by means of these relationships, we will have

$$\left.\begin{aligned} l\frac{\partial u}{\partial l} &= \frac{R^2-l^2}{8\pi\mu}\int_\sigma \frac{L\,d\sigma}{r^3} - \frac{R^2-l^2}{4\pi R}\int_\sigma\left(\frac{\lambda}{2\mu}\xi\theta + \eta\varpi_3 - \zeta\varpi_2\right)\frac{d\sigma}{r^3} + \\ &\quad + \frac{\lambda+\mu}{4\mu}(R^2-l^2)\frac{\partial\theta}{\partial x} + \frac{\lambda+\mu}{16\pi R\mu}(R^2-l^2)\frac{\partial}{\partial x}\int_\sigma \frac{\theta}{x}\,d\sigma, \\ &\cdots\cdots\cdots \end{aligned}\right\} \qquad (38'')$$

because of the first terms on the right sides of equation (38').

It is also convenient to transform these formulas, first noting that /153

$$\begin{aligned} \frac{R^2-l^2}{4\pi R}\int_\sigma \frac{\xi\theta\,d\sigma}{r^3} &= x\frac{R^2-l^2}{4\pi R}\int_\sigma \frac{\theta\,d\sigma}{r^3} + \frac{R^2-l^2}{4\pi R}\int_\sigma \theta\frac{\xi-x}{r^3}\,d\sigma = \\ &= x\theta + \frac{R^2-l^2}{4\pi R}\frac{\partial}{\partial x}\int_\sigma \frac{\theta}{r}\,d\sigma, \\ &\cdots\cdots\cdots \end{aligned}$$

whence

$$\left.\begin{aligned} \frac{R^2-l^2}{4\pi R}\int_\sigma\left(\frac{\lambda}{2\mu}\xi\theta + \eta\varpi_3 - \zeta\varpi_2\right)\frac{d\sigma}{r^3} &= \frac{\lambda}{2\mu}x\theta + y\varpi_3 - z\varpi_2 + \\ + \frac{R^2-l^2}{4\pi R}\left[\frac{\lambda}{2\mu}\frac{\partial}{\partial x}\int_\sigma \frac{\theta}{r}\,d\sigma + \frac{\partial}{\partial y}\int_\sigma \frac{\varpi_3}{r}\,d\sigma - \frac{\partial}{\partial z}\int_\sigma \frac{\varpi_2}{r}\,d\sigma\right]&, \end{aligned}\right\} \qquad (40)$$

Secondarily, among the magnitudes

$$\int_\sigma \frac{\theta}{r}\,d\sigma; \quad \int_\sigma \frac{\varpi_1}{r}\,d\sigma, \quad \int_\sigma \frac{\varpi_2}{r}\,d\sigma, \quad \int_\sigma \frac{\varpi_3}{r}\,d\sigma$$

there are the same differential relationships which exist among the quantities θ; $\bar{\omega}_1$, ω_2, ω_3. In fact, the expression

$$(R^2 - l^2)\left[(\lambda + 2\mu)\frac{\partial}{\partial x}\int_\sigma \frac{\theta}{r} d\sigma + 2\mu\left(\frac{\partial}{\partial z}\int_\sigma \frac{\varpi_2}{r} d\sigma - \frac{\partial}{\partial y}\int_\sigma \frac{\varpi_3}{r} d\sigma\right)\right],$$

for example, becomes zero on σ and its Δ^2 is

$$-2\left[(\lambda + 2\mu)\frac{\partial \theta}{\partial x} + 2\mu\left(\frac{\partial \varpi_2}{\partial z} - \frac{\partial \varpi_3}{\partial y}\right)\right] = 0,$$

which, although harmonic, is identically zero in the entire sphere S and will be

$$(\lambda + 2\mu)\frac{\partial}{\partial x}\int_\sigma \frac{\theta}{r} d\sigma + 2\mu\left(\frac{\partial}{\partial z}\int_\sigma \frac{\varpi_2}{r} d\sigma - \frac{\partial}{\partial y}\int_\sigma \frac{\varpi_3}{r} d\sigma\right) = 0.$$

In short, by means of this relationship, expression (40) reduces to the form

$$\left.\begin{aligned} \frac{R^2 - l^2}{4\pi R}\int_\sigma\left(\frac{\lambda}{2\mu}\xi\theta + \eta\varpi_3 - \zeta\varpi_2\right)\frac{d\sigma}{r^3} = \frac{\lambda}{2\mu}x\theta + y\varpi_3 - z\varpi_2 + \\ + \frac{\lambda + \mu}{4\pi R\mu}(R^2 - l^2)\frac{\partial}{\partial x}\int_\sigma \frac{\theta}{r} d\sigma. \end{aligned}\right\} \quad (40')$$

Thus the equations (38") because of expression (40') and similar formulas become /154

$$\left.\begin{aligned} l\frac{\partial u}{\partial l} &= \frac{R^2 - l^2}{8\pi\mu}\int_\sigma \frac{L\,d\sigma}{r^3} - \frac{\lambda}{2\mu}x\theta - y\varpi_3 + z\varpi_2 + \\ &+ \frac{\lambda + \mu}{4\mu}(R^2 - l^2)\frac{\partial \theta}{\partial x} - \frac{3}{4}\,\frac{\lambda + \mu}{4\pi R\mu}(R^2 - l^2)\frac{\partial}{\partial x}\int_\sigma \frac{\theta}{r} d\sigma, \\ l\frac{\partial v}{\partial l} &= \frac{R^2 - l^2}{8\pi\mu}\int_\sigma \frac{M\,d\sigma}{r^3} - \frac{\lambda}{2\mu}y\theta - z\varpi_1 + x\varpi_3 + \\ &+ \frac{\lambda + \mu}{4\mu}(R^2 - l^2)\frac{\partial \theta}{\partial y} - \frac{3}{4}\,\frac{\lambda + \mu}{4\pi R\mu}(R^2 - l^2)\frac{\partial}{\partial y}\int_\sigma \frac{\theta}{r} d\sigma, \\ l\frac{\partial w}{\partial l} &= \frac{R^2 - l^2}{8\pi\mu}\int_\sigma \frac{N\,d\sigma}{r^3} - \frac{\lambda}{2\mu}z\theta - x\varpi_2 + y\varpi_1 + \\ &+ \frac{\lambda + \mu}{4\mu}(R^2 - l^2)\frac{\partial \theta}{\partial z} - \frac{3}{4}\,\frac{\lambda + \mu}{4\pi R\mu}(R^2 - l^2)\frac{\partial}{\partial z}\int_\sigma \frac{\theta}{r} d\sigma. \end{aligned}\right\} \quad (38'')$$

The solution of our problem is reduced to determining the four unknown functions θ; $\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{\omega}_3$ by the four following equations, of which -- for the sake of brevity -- we write only the first two, since the other two may be immediately derived from the second by circular permutation:

$$\left.\begin{aligned} l\frac{\partial \theta}{\partial l} + \theta = \frac{\partial}{\partial x}\left(\frac{R^2 - l^2}{8\pi\mu}\int_\sigma \frac{L\,d\sigma}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{R^2 - l^2}{8\pi\mu}\int_\sigma \frac{M\,d\sigma}{r^3}\right) + \\ + \frac{\partial}{\partial z}\left(\frac{R^2 - l^2}{8\pi\mu}\int_\sigma \frac{N\,d\sigma}{r^3}\right) - \frac{3\lambda}{2\mu}\theta + \frac{\mu - \lambda}{2\mu}\,l\frac{\partial \theta}{\partial l} + \end{aligned}\right\} \quad (41)$$

$$+\frac{3}{2}\frac{\lambda+\mu}{4\pi R\mu}l\frac{\partial}{\partial l}\int_\sigma \frac{\theta}{r}d\sigma,$$

$$\left.\begin{aligned} l\frac{\partial \varpi_1}{\partial l}+\varpi_1 = \frac{\partial}{\partial y}\left(\frac{R^2-l^2}{16\pi\mu}\int_\sigma \frac{N d\sigma}{r^3}\right)-\frac{\partial}{\partial z}\left(\frac{R^2-l^2}{16\pi\mu}\int_\sigma \frac{M d\sigma}{r^3}\right)+ \\ +\frac{1}{4}\left(z\frac{\partial\theta}{\partial y}-y\frac{\partial\theta}{\partial z}\right)+\frac{l}{2}\frac{\partial\varpi_1}{\partial l}+\varpi_1- \\ -\frac{3}{4}\frac{\lambda+\mu}{4\pi R\mu}\left(z\frac{\partial}{\partial y}\int_\sigma\frac{\theta}{r}d\sigma-y\frac{\partial}{\partial z}\int_\sigma\frac{\theta}{r}d\sigma\right). \end{aligned}\right\} \tag{41}$$

To find θ, let us again use the notation in the preceding section: /155

$$\varphi=\int_\sigma\frac{\theta}{r}d\sigma,\quad 4\pi R\theta=2l\frac{\partial\varphi}{\partial l}+\varphi. \tag{42}$$

With Φ now indicating the expression given by

$$\left.\begin{aligned} \frac{2(\lambda+\mu)}{R}\Phi=\frac{\partial}{\partial x}\left[(R^2-l^2)\int_\sigma\frac{L d\sigma}{r^3}\right]+\frac{\partial}{\partial y}\left[(R^2-l^2)\int_\sigma\frac{M d\sigma}{r^3}\right]+ \\ +\frac{\partial}{\partial z}\left[(R^2-l^2)\int_\sigma\frac{N d\sigma}{r^3}\right], \end{aligned}\right\} \tag{43}$$

the first equation (41) reduces to

$$l^2\frac{\partial^2\varphi}{\partial l^2}+\frac{3\lambda+2\mu}{\lambda+\mu}l\frac{\partial\varphi}{\partial l}+\frac{3\lambda+2\mu}{2(\lambda+\mu)}\varphi=\Phi, \tag{44}$$

or to

$$l\frac{\partial}{\partial l}\left(l\frac{\partial\varphi}{\partial l}+a\varphi\right)+b\left(l\frac{\partial\varphi}{\partial l}+a\varphi\right)=\Phi, \tag{44'}$$

where a and b are the roots of the equation

$$t^2-\frac{2\lambda+\mu}{\lambda+\mu}t+\frac{3\lambda+2\mu}{2(\lambda+\mu)}=0.$$

The values of these two roots are given by

$$a=\frac{2\lambda+\mu+\sqrt{(2\lambda+\mu)^2-2(3\lambda+2\mu)(\lambda+\mu)}}{2(\lambda+\mu)},$$

$$b=\frac{2\lambda+\mu-\sqrt{(2\lambda+\mu)^2-2(3\lambda+2\mu)(\lambda+\mu)}}{2(\lambda+\mu)}$$

and therefore since $\lambda+\mu>0$, $\mu>0$ they are always conjugate imaginary numbers. The general integral of equation (44) or (44') is given by

$$\varphi=\frac{1}{l^a}\int_0^l l^{a-b-1}dl\int_0^l l^{b-1}\Phi\,dl+\frac{\chi_1}{l^a}+\frac{\chi_2}{l^b}, \tag{45}$$

where χ_1 and χ_2 are two arbitrary functions of any two parameters which together with l locate each point in the space. The first part on the right side of equation (45) is a harmonic function, regular in S and real because it does not change value when a and b are interchanged. For the second part to be real, it is necessary that χ_1 and χ_2 be conjugate imaginary numbers. Since /156

$$\frac{\chi_1}{l^a}+\frac{\chi_2}{l^b}=\frac{1}{l^{\frac{a+b}{2}}}\left[(\chi_1+\chi_2)\cos\left(\frac{a-b}{2i}\log l\right)+\frac{\chi_1-\chi_2}{i}\operatorname{sen}\left(\frac{a-b}{2i}\log l\right)\right]$$

may be written and since in this form it is easily verified that the expression is not harmonic it will thus be necessary to set

$$\chi_1=\chi_2=0.$$

Therefore, for ϕ we have

$$\left.\begin{aligned}\varphi&=\frac{1}{l^a}\int_0^l l^{a-b-1}\,d\,l\int_0^l l^{b-1}\,\Phi\,d\,l=\frac{1}{a-b}\left(\frac{1}{l^b}\int_0^l l^{b-1}\,\Phi\,d\,l-\frac{1}{l^a}\int_0^l l^{a-1}\,\Phi\,d\,l\right)=\\&=\frac{-2(\lambda+\mu)}{\sqrt{3(\lambda+\mu)^2-\lambda^2}}\,l^{-\frac{2\lambda+\mu}{2(\lambda+\mu)}}\int_0^l\tau^{-\frac{\mu}{2(\lambda+\mu)}}\operatorname{sen}\left(\frac{\sqrt{3(\lambda+\mu)^2-\lambda^2}}{2(\lambda+\mu)}\log\frac{\tau}{l}\right)\Phi\,d\,\tau.\end{aligned}\right\}\quad(45')$$

After ϕ has been found, and therefore θ by means of expression (42), the rotations $\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{\omega}_3$ are calculated by simple quadrature. The second equation (41) may in fact be written

$$\left.\begin{aligned}l\frac{\partial\bar{\omega}_1}{\partial l}&=\frac{\partial}{\partial y}\left(\frac{R^2-l^2}{8\pi\mu}\int_\sigma\frac{N\,d\,\sigma}{r^3}\right)-\frac{\partial}{\partial z}\left(\frac{R^2-l^2}{8\pi\mu}\int_\sigma\frac{M\,d\,\sigma}{r^3}\right)+\\&+\frac{l}{4\pi R}\frac{\partial}{\partial l}\left(z\frac{\partial\varphi}{\partial y}-y\frac{\partial\varphi}{\partial z}\right)-\frac{3\lambda+2\mu}{8\pi R\mu}\left(z\frac{\partial\varphi}{\partial y}-y\frac{\partial\varphi}{\partial z}\right).\end{aligned}\right\}\quad(46)$$

When this equation is divided by l, integrated between 0 and l, and when it is noted that the arbitrary quantity introduced by integration -- since it must be a harmonic function regular in S and independent of l -- cannot be reduced to anything but a constant h_1, we find

$$\left.\begin{aligned}\bar{\omega}_1&=\int_0^l\frac{d\,l}{l}\left[\frac{\partial}{\partial y}\left(\frac{R^2-l^2}{8\pi\mu}\int_\sigma\frac{N\,d\,\sigma}{r^3}\right)-\frac{\partial}{\partial z}\left(\frac{R^2-l^2}{8\pi\mu}\int_\sigma\frac{M\,d\,\sigma}{r^3}\right)\right]+\\&+\frac{1}{4\pi R}\left(z\frac{\partial\varphi}{\partial y}-y\frac{\partial\varphi}{\partial z}\right)-\frac{3\lambda+2\mu}{8\pi R\mu}\int_0^l\left(\frac{z}{l}\frac{\partial\varphi}{\partial y}-\frac{y}{l}\frac{\partial\varphi}{\partial z}\right)d\,l+h_1.\end{aligned}\right\}\quad(47)$$

If it is noted that, because of the relationship

$$\int_\sigma(N\eta-M\zeta)\,d\,\sigma=0,$$

we may write /157

$$\frac{\partial}{\partial y}\left(\frac{R^2-l^2}{8\pi\mu}\int_\sigma\frac{N\,d\,\sigma}{r^3}\right)-\frac{\partial}{\partial z}\left(\frac{R^2-l^2}{8\pi\mu}\int_\sigma\frac{M\,d\,\sigma}{r^3}\right)=$$

$$= -\frac{y}{8\pi\mu}\int_\sigma N\left(\frac{2}{r^3}+3\frac{R^2-l^2}{r^5}\right)d\sigma + \frac{z}{8\pi\mu}\int_\sigma M\left(\frac{2}{r^3}+3\frac{R^2-l^2}{r^5}\right)d\sigma +$$

$$+3\frac{R^2-l^2}{8\pi\mu}\int_\sigma (N\eta - M\zeta)\left(\frac{1}{r^5}-\frac{1}{R^5}\right)d\sigma$$

it immediately results that the first term on the right side of expression (47) is also finite for $l = 0$.

In similar fashion $\tilde{\omega}_2$ and $\tilde{\omega}_3$ are calculated, and the relative expressions are obtained by expression (47) by cyclic interchange of the indices 1, 2, 3 and the letters x, y, z; L, M, N.

Displacements u, v, w which still remain to be determined are obtained from expression (38''') by quadrature. If we divide each of the equations (38''') by l, integrate between 0 and l, and note that the arbitrary quantities introduced by integration can be reduced only to constants k_1, k_2, k_3 since -- like the other parts of u, v, w, -- they must be functions regular in S and satisfy equation $\Delta^2\Delta^2 = 0$, as well as be independent of l, we immediately find

$$\left.\begin{aligned}
u &= \int_0^l \frac{dl}{l}\left\{\frac{R^2-l^2}{8\pi\mu}\int_\sigma \frac{L\,d\sigma}{r^3} - \frac{\lambda}{2\mu}x\theta - y\tilde{\omega}_3 + z\tilde{\omega}_2 + \right.\\
&\left. + \frac{\lambda+\mu}{4\mu}(R^2-l^2)\frac{\partial\theta}{\partial x} - \frac{3}{4}\,\frac{\lambda+\mu}{4\pi R\mu}(R^2-l^2)\frac{\partial\varphi}{\partial x}\right\} + k_1 - h_3 y + h_2 z,\\
v &= \int_0^l \frac{dl}{l}\left\{\frac{R^2-l^2}{8\pi\mu}\int_\sigma \frac{M\,d\sigma}{r^3} - \frac{\lambda}{2\mu}y\theta - z\tilde{\omega}_1 + x\tilde{\omega}_3 + \right.\\
&\left. + \frac{\lambda+\mu}{4\mu}(R^2-l^2)\frac{\partial\theta}{\partial y} - \frac{3}{4}\,\frac{\lambda+\mu}{4\pi R\mu}(R^2-l^2)\frac{\partial\varphi}{\partial y}\right\} + k_2 - h_1 z + h_3 x,\\
w &= \int_0^l \frac{dl}{l}\left\{\frac{R^2-l^2}{8\pi\mu}\int_\sigma \frac{N\,d\sigma}{r^3} - \frac{\lambda}{2\mu}z\theta - x\tilde{\omega}_2 + y\tilde{\omega}_1 + \right.\\
&\left. + \frac{\lambda+\mu}{4\mu}(R^2-l^2)\frac{\partial\theta}{\partial z} - \frac{3}{4}\,\frac{\lambda+\mu}{4\pi R\mu}(R^2-l^2)\frac{\partial\varphi}{\partial z}\right\} + k_3 - h_2 x + h_1 y.
\end{aligned}\right\}\quad (48)$$

Here, too, if we note that, because of the relationships, /158

$$\int_\sigma L\,d\sigma = 0, \ldots$$

we may write

$$\int_0^l \frac{dl}{l}\frac{R^2-l^2}{8\pi\mu}\int_\sigma \frac{L\,d\sigma}{r^3} = \int_0^l \frac{dl}{l}\frac{R^2-l^2}{4\pi\mu}\int_\sigma L\left(\frac{1}{r^3}-\frac{1}{R^3}\right)d\sigma, \ldots$$

and that

$$\frac{\partial\theta}{\partial x} - \frac{3}{4\pi R}\frac{\partial\varphi}{\partial x} = \frac{l}{2\pi R}\frac{\partial}{\partial l}\frac{\partial\varphi}{\partial x}, \ldots$$

it results that expressions (48) for u, v, w are also finite for $l = 0$.

Conversely, if L, M, N are finite and continuous functions of the coordinates of the points of sphere σ together with the first derivatives and they have finite second derivatives, the Φ determined by (43), the ϕ determined by expression (45'), and the θ given by the second equation in expression (42) by means of ϕ are harmonic functions regular in S and tend toward finite values in the σ points, together with the first derivatives. This occurs because, for example

$$\frac{\partial}{\partial x}\left[(R^2-l^2)\int_\sigma \frac{L\,d\sigma}{r^3}\right]=2l\frac{\partial}{\partial l}\frac{\partial}{\partial x}\int_\sigma \frac{L\,d\sigma}{r}+3\frac{\partial}{\partial x}\int_\sigma \frac{L\,d\sigma}{r}$$

and because of the well-known properties of the potential function. Similarly, $\overline{\omega}_1$, $\overline{\omega}_2$, $\overline{\omega}_3$ given by expression (47) and similar equations are functions harmonic and regular in S which tend toward finite values in σ points. Since the equations

$$\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z};\qquad \varpi_1=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\ldots$$

are evidently satisfied, if expression (2) is satisfied for the θ; $\overline{\omega}_1$, $\overline{\omega}_2$, $\overline{\omega}_3$ expression (1) is also satisfied. To show that the values of u, v, w which we have found also satisfy expression (1), it therefore suffices to demonstrate the fact that the corresponding values of θ; $\overline{\omega}_1$, $\overline{\omega}_2$, $\overline{\omega}_3$ satisfy expression (2). In the meantime,

$$\frac{\partial}{\partial z}\left(l\frac{\partial \varpi_2}{\partial l}\right)-\frac{\partial}{\partial y}\left(l\frac{\partial \varpi_3}{\partial l}\right)=-\frac{\lambda+\mu}{4\pi R\mu}\frac{\partial \Phi}{\partial x}-$$
$$-\frac{1}{4\pi R}\frac{\partial}{\partial x}\left[l^2\frac{\partial^2 \varphi}{\partial l^2}+\frac{2\mu-3\lambda}{2\mu}l\frac{\partial \varphi}{\partial l}+\frac{3\lambda+2\mu}{2\mu}\varphi\right]$$

is easily derived analogously to expression (46). However, on the other hand, by expression (42) /159

$$\frac{\lambda+2\mu}{2\mu}\frac{\partial}{\partial x}\left(l\frac{\partial \theta}{\partial l}\right)=\frac{1}{4\pi R}\frac{\partial}{\partial x}\left[\frac{\lambda+2\mu}{\mu}l^2\frac{\partial^2 \varphi}{\partial l^2}+\frac{3}{2}\frac{\lambda+2\mu}{\mu}l\frac{\partial \varphi}{\partial l}\right],$$

hence,

$$\frac{\partial}{\partial z}\left(l\frac{\partial \varpi_2}{\partial l}\right)-\frac{\partial}{\partial y}\left(l\frac{\partial \varpi_3}{\partial l}\right)+\frac{\lambda+2\mu}{2\mu}\frac{\partial}{\partial x}\left(l\frac{\partial \theta}{\partial l}\right)=$$
$$=\frac{\lambda+\mu}{4\pi R\mu}\frac{\partial}{\partial x}\left[-\Phi+l^2\frac{\partial^2 \varphi}{\partial l^2}+\frac{3\lambda+2\mu}{\lambda+\mu}l\frac{\partial \varphi}{\partial l}+\frac{3\lambda+2\mu}{2(\lambda+\mu)}\varphi\right]=0,$$

because of expression (44). Integrating the equation

$$\frac{\partial}{\partial z}\left(l\frac{\partial \varpi_2}{\partial l}\right)-\frac{\partial}{\partial y}\left(l\frac{\partial \varpi_3}{\partial l}\right)+\frac{\lambda+2\mu}{2\mu}\frac{\partial}{\partial x}\left(l\frac{\partial \theta}{\partial l}\right)=$$
$$=\frac{\partial}{\partial l}\left\{l\left[\frac{\partial \varpi_2}{\partial z}-\frac{\partial \varpi_3}{\partial y}+\frac{\lambda+2\mu}{2\mu}\frac{\partial \theta}{\partial x}\right]\right\}=0$$

with respect to l on condition that $\frac{\partial \varpi_2}{\partial z}-\frac{\partial \varpi_3}{\partial y}+\frac{\lambda+2\mu}{2\mu}\frac{\partial \theta}{\partial x}$ be regular in S, it is found that

$$\frac{\partial \varpi_2}{\partial z} - \frac{\partial \varpi_3}{\partial y} + \frac{\lambda + 2\mu}{2\mu}\frac{\partial \theta}{\partial x} = 0$$

or expression (2) is satisfied. Finally, since from expression (38''') it follows that

$$\lambda\,\theta\frac{x}{l} + 2\mu\left(\frac{\partial u}{\partial l} + \varpi_3\frac{y}{l} - \varpi_2\frac{z}{l}\right) = \frac{R^2 - l^2}{4\pi l}\int_\sigma \frac{L\,d\sigma}{r^3} +$$

$$+ \frac{\lambda + \mu}{4\pi R}(R^2 - l^2)\frac{\partial}{\partial l}\frac{\partial \varphi}{\partial x},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

and $\frac{\partial}{\partial l}\frac{\partial \varphi}{\partial x}, \ldots$ tend toward finite values when l = R, the surface conditions are also satisfied.

The indicated method of solving the preceding problem holds true whatever the values of λ and μ. However, if $\lambda + \mu > 0$, $\mu > 0$, we may also state that the problem has a single determinate solution, unless there is an arbitrary rigid displacement of the elastic body.

2. *External space.* The problem of elastic equilibrium for the space external to the spherical surface of radius R, when L, M, N are given on the surface, may be solved by a method entirely analogous to the preceding. Taking our start from expression (37), after having applied formulas (34) to it, and the conversion formula /160

$$\int_\sigma \frac{\partial \theta}{\partial \rho}\frac{d\sigma}{r} = \int_\sigma \theta\frac{\partial \frac{1}{r}}{\partial \rho}\,d\sigma - 4\pi\theta = -2\pi\theta - \frac{1}{2R}\int_\sigma \frac{\theta}{r}\,d\sigma,$$

we find

$$l\frac{\partial u}{\partial l} = \frac{l^2 - R^2}{4\pi}\int_\sigma \frac{\partial u}{\partial \rho}\frac{d\sigma}{r^3} - \frac{\lambda + \mu}{4\mu}(l^2 - R^2)\frac{\partial \theta}{\partial x} +$$

$$+ \frac{\lambda + \mu}{16\pi R\mu}(l^2 - R^2)\frac{\partial}{\partial x}\int_\sigma \frac{\theta}{r}\,d\sigma,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Since now on σ

$$\cos nx = \frac{x}{R}, \qquad \cos ny = \frac{y}{R}, \qquad \cos nz = \frac{z}{R}; \qquad \frac{d}{dn} = \frac{\partial}{\partial l},$$

the boundary conditions give us

$$\frac{\partial u}{\partial l} = -\frac{L}{2\mu} - \frac{\lambda}{2\mu}\frac{x}{R}\theta - \frac{y}{R}\varpi_3 + \frac{z}{R}\varpi_2, \ldots$$

and therefore we may write:

$$l\frac{\partial u}{\partial l}=-\frac{l^2-R^2}{8\pi\mu}\int_\sigma\frac{L\,d\sigma}{r^3}-\frac{l^2-R^2}{4\pi R}\int_\sigma\left(\frac{\lambda}{2\mu}\xi\theta+\eta\varpi_3-\zeta\varpi_2\right)\frac{d\sigma}{r^3}-$$

$$-\frac{\lambda+\mu}{4\mu}(l^2-R^2)\frac{\partial\theta}{\partial x}+\frac{\lambda+\mu}{16\pi R\mu}(l^2-R^2)\frac{\partial}{\partial x}\int_\sigma\frac{\theta}{r}d\sigma,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Noting then that

$$\frac{l^2-R^2}{4\pi R}\int_\sigma\frac{\xi\theta}{r^3}d\sigma=x\theta+\frac{l^2-R^2}{4\pi R}\frac{\partial}{\partial x}\int_\sigma\frac{\theta}{r}d\sigma,\ldots$$

we find

$$\frac{l^2-R^2}{4\pi R}\int_\sigma\left(\frac{\lambda}{2\mu}\xi\theta+\eta\tilde\omega_3-\zeta\tilde\omega_2\right)\frac{d\sigma}{r^3}=\frac{\lambda}{2\mu}x\theta+y\tilde\omega_3-z\tilde\omega_2+$$

$$+\frac{l^2-R^2}{4\pi R}\left(\frac{\lambda}{2\mu}\frac{\partial}{\partial x}\int_\sigma\frac{\theta}{r}d\sigma+\frac{\partial}{\partial y}\int_\sigma\frac{\tilde\omega_3}{r}d\sigma-\frac{\partial}{\partial z}\int_\sigma\frac{\tilde\omega_2}{r}d\sigma\right)=$$

$$=\frac{\lambda}{2\mu}x\theta+y\tilde\omega_3-z\tilde\omega_2+\frac{\lambda+\mu}{4\pi R\mu}(l^2-R^2)\frac{\partial}{\partial x}\int_\sigma\frac{\theta}{r}d\sigma,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and hence in place of expression (38''') we will have the formulas /161

$$l\frac{\partial u}{\partial l}=-\frac{l^2-R^2}{8\pi\mu}\int_\sigma\frac{L\,d\sigma}{r^3}-\frac{\lambda}{2\mu}x\theta-y\tilde\omega_3+z\tilde\omega_2-$$

$$-\frac{\lambda+\mu}{4\mu}(l^2-R^2)\frac{\partial\theta}{\partial x}-\frac{3}{4}\frac{\lambda+\mu}{4\pi R\mu}(l^2-R^2)\frac{\partial}{\partial x}\int_\sigma\frac{\theta}{r}d\sigma,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

These equations are identical to those in expression (38'''), except in the sign of the last term. In similar fashion, the equations which θ; $\tilde\omega_1$, $\tilde\omega_2$, $\tilde\omega_3$ must satisfy will be identical to expression (41), except for the sign of the last term. If now, still setting

$$\varphi=\int_\sigma\frac{\theta}{r}d\sigma,$$

we note that

$$-4\pi R\theta=2l\frac{\partial\varphi}{\partial l}+\varphi, \tag{50}$$

we immediately find that ϕ satisfies the same differential equation (44) when it is assumed that Φ is determined by the relationship

$$\frac{2(\lambda+\mu)}{R}\Phi = \frac{\partial}{\partial x}\left[(l^2-R^2)\int_\sigma \frac{L\,d\sigma}{r^3}\right] + \frac{\partial}{\partial y}\left[(l^2-R^2)\int_\sigma \frac{M\,d\sigma}{r^3}\right] + \frac{\partial}{\partial z}\left[(l^2-R^2)\int_\sigma \frac{N\,d\sigma}{r^3}\right]. \tag{51}$$

We will therefore still be able to set

$$\varphi = \frac{-2(\lambda+\mu)}{\sqrt{3(\lambda+\mu)^2-\lambda^2}}\, l^{-\frac{2\lambda+\mu}{2(\lambda+\mu)}} \int_\infty^l \tau^{-\frac{\mu}{2(\lambda+\mu)}} \operatorname{sen}\left(\frac{\sqrt{3(\lambda+\mu)^2-\lambda^2}}{2(\lambda+\mu)}\log\frac{\tau}{l}\right)\Phi\,d\tau. \tag{52}$$ /162

Solution of the problem is completed by the formulas

$$\tilde{\omega}_1 = -\int_\infty^l \frac{dl}{l}\left[\frac{\partial}{\partial y}\left(\frac{l^2-R^2}{8\pi\mu}\int_\sigma \frac{N\,d\sigma}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{l^2-R^2}{8\pi\mu}\int \frac{M\,d\sigma}{r^3}\right)\right] - \frac{1}{4\pi R}\left(z\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial z}\right) + \frac{3\lambda+2\mu}{8\pi R\mu}\int_\infty^l\left(\frac{z}{l}\frac{\partial\varphi}{\partial y} - \frac{y}{l}\frac{\partial\varphi}{\partial z}\right)dl, \tag{53}$$

. . . .

$$u = \int_\infty^l \frac{dl}{l}\left[-\frac{l^2-R^2}{8\pi\mu}\int \frac{L\,d\sigma}{r^3} - \frac{\lambda}{2\mu}x\theta - y\tilde{\omega}_3 + z\tilde{\omega}_2 - \frac{\lambda+\mu}{4\mu}(l^2-R^2)\frac{\partial\theta}{\partial x} - \frac{3}{4}\frac{\lambda+\mu}{4\pi R\mu}(l^2-R^2)\frac{\partial\varphi}{\partial x}\right], \tag{54}$$

. . . .

Functions Φ, ϕ, and θ given by expressions (51), (52), and (50) become zero at infinity, and the other functions $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{\omega}_3$; u, v, w have been determined on that condition. An a posteriori check is made with considerations similar to those made in the preceding case.

3. *Case in which L, v, w are given on the limiting sphere.* Because of the symmetry of the spherical surface with regard to the coordinate axes, the problems in which u, M, w or u, v, N are given on the sphere do not differ from the problem which we propose to solve, except for the different name of the coordinate axes. In order not to dwell too long on these new problems, we will disregard the case of external space, because it is easy to make the extension and these problems do not have the importance of those already solved for elasticity theory.

We will, therefore, begin first of all to establish certain notation and certain formulas which we will constantly use, i.e., let us set

$$\varphi = \int_\sigma \frac{\theta}{r}d\sigma, \quad \varphi' = \frac{1}{l}\int_0^l dl\int_\sigma \frac{\theta}{r}d\sigma,$$
$$\psi_i = \int_\sigma \frac{\tilde{\omega}_i}{r}d\sigma, \quad \psi'_i = \frac{1}{l}\int_0^l dl\int_\sigma \frac{\tilde{\omega}_i}{r}d\sigma, \quad i = 1, 2, 3. \tag{55}$$

ϕ, ϕ'; ψ_i, ψ'_i are harmonic functions, regular in S and among them the following relationships hold:

$$\left.\begin{aligned} \varphi &= \frac{\partial}{\partial l}(l\varphi') = \varphi' + l\frac{\partial \varphi'}{\partial l}, \quad \varphi' = \frac{1}{l}\int_0^l \varphi\, dl; \\ \psi_i &= \frac{\partial}{\partial l}(l\psi'_i) = \psi'_i + l\frac{\partial \psi'_i}{\partial l}, \quad \psi'_i = \frac{1}{l}\int_0^l \psi_i\, dl \end{aligned}\right\} \tag{56}$$

This system of formulas is completed by the formulas connecting ϕ and ψ_i and θ and $\tilde{\omega}_i$, i.e., /163

$$\left.\begin{aligned} 4\pi R\theta &= 2l\frac{\partial \varphi}{\partial l} + \varphi, \quad \varphi = 2\pi R l^{-\frac{1}{2}}\int_0^l l^{-\frac{1}{2}}\theta\, dl; \\ 4\pi R\tilde{\omega}_i &= 2l\frac{\partial \psi_i}{\partial l} + \psi_i, \quad \psi_i = 2\pi R l^{-\frac{1}{2}}\int_0^l l^{-\frac{1}{2}}\tilde{\omega}_i\, dl. \end{aligned}\right\} \tag{56}$$

It is easily seen that ϕ'; ψ'_1, ψ'_2, ψ'_3 satisfy the same differential relationships satisfied by ϕ; ψ_1, ψ_2, ψ_3 and therefore θ; $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{\omega}_3$. We have, for example,

$$(\lambda + 2\mu)\frac{\partial \varphi'}{\partial x} + 2\mu\left(\frac{\partial \psi'_2}{\partial z} - \frac{\partial \psi'_3}{\partial y}\right) = \\ = \frac{1}{l^2}\int_0^l l\left[(\lambda + 2\mu)\frac{\partial \varphi}{\partial x} + 2\mu\left(\frac{\partial \psi_2}{\partial z} - \frac{\partial \psi_3}{\partial y}\right)\right] = 0\,(*).$$

For purpose of brevity, let us also set /164

$$U = (R^2 - l^2)\int_\sigma \frac{u\, d\sigma}{r^3}, \quad V = (R^2 - l^2)\int_\sigma \frac{v\, d\sigma}{r^3}, \quad W = (R^2 - l^2)\int_\sigma \frac{w\, d\sigma}{r^3}; \tag{57}$$

---

* More generally the expressions

$$\frac{1}{l^c}\int_0^l l^{c-1}\theta\, dl; \ \frac{1}{l^c}\int_0^l l^{c-1}\tilde{\omega}_1\, dl, \ \frac{1}{l^c}\int_0^l l^{c-1}\tilde{\omega}_2\, dl, \ \frac{1}{l^c}\int_0^l l^{c-1}\tilde{\omega}_3\, dl$$

where c is any positive constant, are harmonic, regular in S, and related by the same differential relations by which θ; $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{\omega}_3$ are related. The same may be said of the expressions which are obtained by performing operation $\frac{1}{l^c}\int_0^l l^{c-1}\ldots dl$ on θ; $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{\omega}_3$ any number of times, the value of constant c being variable each time the indicated operation is applied.

$$\left.\begin{aligned}
\mathfrak{L} &= \int_\sigma L\left(\frac{1}{r} - \frac{1}{2R}\log\frac{R - l\cos\omega + r}{2R}\right)d\sigma,\\
\mathfrak{M} &= \int_\sigma M\left(\frac{1}{r} - \frac{1}{2R}\log\frac{R - l\cos\omega + r}{2R}\right)d\sigma,\\
\mathfrak{N} &= \int_\sigma N\left(\frac{1}{r} - \frac{1}{2R}\log\frac{R - l\cos\omega + r}{2R}\right)d\sigma,
\end{aligned}\right\} \tag{57}$$

and let us note that

$$l\frac{\partial \mathfrak{L}}{\partial l} = \frac{R^2 - l^2}{2}\int_\sigma \frac{L}{r^3}\,d\sigma - \frac{1}{2R}\int_\sigma L\,d\sigma = \frac{R^2 - l^2}{2}\int_\sigma \frac{L}{r^3}\,d\sigma,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

if, as we assume, conditions

$$\int_\sigma L\,d\sigma = 0, \ldots$$

are satisfied.

After this, in order to solve the problem we will make use of the first of formulas (38'''), divided by l and integrated from 0 to l, and of the last two formulas (30):

$$\left.\begin{aligned}
u &= \frac{\mathfrak{L}}{4\pi\mu} - \frac{1}{2\pi R}\left(\frac{\lambda}{2\mu}x\varphi + y\psi_3 - z\psi_2\right) + \frac{1}{4\pi R}\left(\frac{\lambda}{2\mu}x\varphi' + y\psi'_3 - z\psi'_2\right) +\\
&\quad + \frac{\lambda+\mu}{4\pi R\mu}l^2\frac{\partial\varphi'}{\partial x} + \frac{\lambda+\mu}{8\pi R\mu}(R^2 - l^2)\frac{\partial\varphi}{\partial x} + \text{cost. (*)},\\
v &= \frac{V}{4\pi R} + \frac{\lambda+\mu}{8\pi R\mu}(R^2 - l^2)\frac{\partial\varphi}{\partial y},\\
w &= \frac{W}{4\pi R} + \frac{\lambda+\mu}{8\pi R\mu}(R^2 - l^2)\frac{\partial\varphi}{\partial z}.
\end{aligned}\right\} \tag{58}$$

The problem is reduced to determining the unknown functions θ; $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{\omega}_3$ of equations /165

$$\begin{aligned}
\theta &= \frac{1}{4\pi\mu}\frac{\partial\mathfrak{L}}{\partial x} + \frac{1}{4\pi R}\left(\frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right) - \frac{\lambda+\mu}{4\pi R\mu}l\frac{\partial\varphi}{\partial l} +\\
&\quad + \frac{\lambda+\mu}{4\pi R\mu}l^2\frac{\partial^2\varphi'}{\partial x^2} + \frac{\lambda+\mu}{2\pi R\mu}x\frac{\partial\varphi'}{\partial x} - \frac{1}{2\pi R}\left[\frac{\lambda}{2\mu}\left(\varphi + x\frac{\partial\varphi}{\partial x}\right) +\right.
\end{aligned} \tag{59}$$

---

* Formulas of this sort could have been derived by starting from expression (8) and substituting expression

$$G_1 = \frac{1}{r} + \frac{R}{l}\frac{1}{r_1} - \int_0^l \frac{1}{\rho}\left(\frac{1}{R} - \frac{R}{l}\frac{1}{r_1}\right)d\rho$$

therein for G_1. With the formulas to which we are referring, it would also have been possible to solve the problem in Section 2, but these formulas are less susceptible to easy transformations than are those which we preferred.

$$
\left.\begin{aligned}
&+y\frac{\partial\psi_3}{\partial x}-z\frac{\partial\psi_2}{\partial x}\Big]+\frac{1}{4\pi R}\Big[\frac{\lambda}{2\mu}\Big(\varphi'+x\frac{\partial\varphi'}{\partial x}\Big)+y\frac{\partial\psi'_3}{\partial x}-z\frac{\partial\psi'_2}{\partial x}\Big],\\
2\bar{\omega}_1&=\frac{1}{4\pi R}\Big(\frac{\partial W}{\partial y}-\frac{\partial V}{\partial z}\Big)-\frac{\lambda+\mu}{4\pi R\mu}\Big(y\frac{\partial\varphi}{\partial z}-z\frac{\partial\varphi}{\partial y}\Big),\\
2\bar{\omega}_2&=\frac{1}{4\pi\mu}\frac{\partial\varrho}{\partial z}-\frac{1}{4\pi R}\frac{\partial W}{\partial x}-\frac{\lambda+\mu}{4\pi R\mu}\Big(z\frac{\partial\varphi}{\partial x}-x\frac{\partial\varphi}{\partial z}\Big)+\\
&+\frac{\lambda+\mu}{4\pi R\mu}l^2\frac{\partial^2\varphi'}{\partial x\,\partial z}+\frac{\lambda+\mu}{2\pi R\mu}z\frac{\partial\varphi'}{\partial x}-\\
&-\frac{1}{2\pi R}\Big[\frac{\lambda}{2\mu}x\frac{\partial\varphi}{\partial z}+y\frac{\partial\psi_3}{\partial z}-z\frac{\partial\psi_2}{\partial z}-\psi_2\Big]+\\
&+\frac{1}{4\pi R}\Big[\frac{\lambda}{2\mu}x\frac{\partial\varphi'}{\partial z}+y\frac{\partial\psi'_3}{\partial z}-z\frac{\partial\psi'_2}{\partial z}-\psi'_2\Big],\\
2\bar{\omega}_3&=\frac{1}{4\pi R}\frac{\partial V}{\partial x}-\frac{1}{4\pi\mu}\frac{\partial\varrho}{\partial y}-\frac{\lambda+\mu}{4\pi R\mu}\Big(x\frac{\partial\varphi}{\partial y}-y\frac{\partial\varphi}{\partial x}\Big)-\\
&-\frac{\lambda+\mu}{4\pi R\mu}l^2\frac{\partial^2\varphi'}{\partial x\,\partial y}-\frac{\lambda+\mu}{2\pi R\mu}y\frac{\partial\varphi'}{\partial x}+\\
&+\frac{1}{2\pi R}\Big[\frac{\lambda}{2\mu}x\frac{\partial\varphi}{\partial y}+y\frac{\partial\psi_3}{\partial y}+\psi_3-z\frac{\partial\psi_2}{\partial y}\Big]-\\
&-\frac{1}{4\pi R}\Big[\frac{\lambda}{2\mu}x\frac{\partial\varphi'}{\partial y}+y\frac{\partial\psi'_3}{\partial y}+\psi'_3-z\frac{\partial\psi'_2}{\partial y}\Big].
\end{aligned}\right\}\qquad(59)
$$

In order to do this, let us meantime note that the last two terms of the first of these equations, except for the sign, may be written

$$
\left.\begin{aligned}
&\frac{\lambda}{8\pi R\mu}\Big[2\varphi-\varphi'+x\frac{\partial}{\partial x}(2\varphi-\varphi')\Big]+\\
&+\frac{1}{4\pi R}\Big[y\frac{\partial}{\partial x}(2\psi_3-\psi'_3)-z\frac{\partial}{\partial x}(2\psi_2-\psi'_2)\Big]
\end{aligned}\right\}\qquad(60)
$$

and since

$$
\left.\begin{aligned}
\frac{\partial}{\partial x}(2\psi_3-\psi'_3)&=-\frac{\lambda+2\mu}{2\mu}\frac{\partial}{\partial y}(2\varphi-\varphi')+\frac{\partial}{\partial z}(2\psi_1-\psi'_1),\\
\frac{\partial}{\partial x}(2\psi_2-\psi'_2)&=\frac{\lambda+2\mu}{2\mu}\frac{\partial}{\partial z}(2\varphi-\varphi')+\frac{\partial}{\partial y}(2\psi_1-\psi'_1),\\
2\psi_1-\psi'_1&=\frac{4\pi R}{l}\int_0^l\bar{\omega}_1\,dl_1
\end{aligned}\right\}\qquad(61)
$$

the total of terms (60) reduces to /166

$$
\left.\begin{aligned}
&\frac{\lambda}{8\pi R\mu}(2\varphi-\varphi')+\frac{\lambda+\mu}{4\pi R\mu}x\frac{\partial}{\partial x}(2\varphi-\varphi')-\frac{\lambda+2\mu}{8\pi R\mu}l\frac{\partial}{\partial l}(2\varphi-\varphi')+\\
&+y\frac{\partial}{\partial z}\Big(\frac{1}{l}\int_0^l\bar{\omega}_1\,dl\Big)-z\frac{\partial}{\partial y}\Big(\frac{1}{l}\int_0^l\bar{\omega}_1\,dl\Big)=\frac{\lambda}{8\pi R\mu}(2\varphi-\varphi')+
\end{aligned}\right\}\qquad(60')
$$

$$
\left.\begin{aligned}
&+\frac{\lambda+\mu}{4\pi R\mu}x\frac{\partial}{\partial x}(2\varphi-\varphi')-\frac{\lambda+2\mu}{8\pi R\mu}l\frac{\partial}{\partial l}(2\varphi-\varphi')-\\
&\qquad-\frac{\lambda+\mu}{8\pi R\mu}\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)^2\varphi'+\\
&+\frac{1}{8\pi Rl}\int_0^l dl\left[y\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial y}-\frac{\partial V}{\partial z}\right)-z\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial y}-\frac{\partial V}{\partial z}\right)\right].
\end{aligned}\right\}\qquad (60')
$$

We obtain the last transformation by substituting the value given by the second of equations (59) for $\tilde{\omega}_1$, and have indicated by $\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)^2$ the operation which consists of performing operation $y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}$ twice.

We may therefore replace the first of equations (59) with the other one:

$$
\left.\begin{aligned}
&\frac{\lambda+2\mu}{2\mu}\varphi'+\frac{3\lambda+8\mu}{2\mu}l\frac{\partial\varphi'}{\partial l}+l^2\frac{\partial^2\varphi'}{\partial l^2}-\frac{\lambda+\mu}{\mu}x\frac{\partial\varphi'}{\partial x}+\\
&+2\frac{\lambda+\mu}{\mu}x\frac{\partial}{\partial x}\left(l\frac{\partial\varphi'}{\partial l}\right)-\frac{\lambda+\mu}{\mu}l^2\frac{\partial^2\varphi'}{\partial x^2}-\frac{\lambda+\mu}{2\mu}\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)^2\varphi'=\\
&\qquad=\frac{R}{\mu}\frac{\partial\varrho}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}-\\
&-\frac{1}{2l}\int_0^l dl\left[y\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial y}-\frac{\partial V}{\partial z}\right)-z\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial y}-\frac{\partial V}{\partial z}\right)\right].
\end{aligned}\right\}\qquad (62)
$$

Observing now that

$$
\begin{aligned}
&-x\frac{\partial\varphi'}{\partial x}+2x\frac{\partial}{\partial x}\left(l\frac{\partial\varphi'}{\partial l}\right)-l^2\frac{\partial^2\varphi'}{\partial x^2}=\\
&=x\frac{\partial\varphi'}{\partial x}+(x^2-y^2-z^2)\frac{\partial^2\varphi'}{\partial x^2}+2xy\frac{\partial^2\varphi'}{\partial x\,\partial y}+2xz\frac{\partial^2\varphi'}{\partial x\,\partial z}=\\
&=\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)^2\varphi'+l^2\frac{\partial^2\varphi'}{\partial l^2}+l\frac{\partial\varphi'}{\partial l}.
\end{aligned}
$$

expression (62) is transformed into the other /167

$$
\left.\begin{aligned}
&\frac{\lambda+2\mu}{2\mu}\varphi'+\frac{5}{2}\frac{\lambda+2\mu}{\mu}l\frac{\partial\varphi'}{\partial l}+\frac{\lambda+2\mu}{\mu}l^2\frac{\partial^2\varphi'}{\partial l^2}+\\
&+\frac{\lambda+\mu}{2\mu}\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)^2\varphi'=\frac{R}{\mu}\frac{\partial\varrho}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}-\\
&-\frac{1}{2l}\int_0^l dl\left[y\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial y}-\frac{\partial V}{\partial z}\right)-z\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial y}-\frac{\partial V}{\partial z}\right)\right].
\end{aligned}\right\}\qquad (62')
$$

From this equation, ϕ' must first of all be derived. For this purpose, it is well to introduce polar coordinates, for example, setting

$$x = l\cos\alpha, \qquad y = l\,\mathrm{sen}\,\alpha\cos\beta, \qquad z = l\,\mathrm{sen}\,\alpha\,\mathrm{sen}\,\beta.$$

It is then immediately found that

$$y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} = \frac{\partial}{\partial\beta}$$

and therefore expression (62') is transformed into the other

$$\left.\begin{aligned}\frac{1}{2}\varphi' + \frac{5}{2}l\frac{\partial\varphi'}{\partial l} + l^2\frac{\partial\varphi'}{\partial l^2} + \frac{\lambda+\mu}{2(\lambda+2\mu)}\frac{\partial^2\varphi'}{\partial\beta^2} = \frac{\mu}{\lambda+2\mu}\Big\{\frac{R}{\mu}\frac{\partial\varrho}{\partial x} +\\ + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} - \frac{1}{2l}\int_0^l dl\left[y\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z}\right) - z\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z}\right)\right]\Big\}.\end{aligned}\right\} \quad (62'')$$

If the deformation comes from rotation about the x-axis, ϕ' is determined by quadratures.

In the general case, this equation, taking log l for the independent variable instead of l, becomes an equation with constant coefficients of the elliptic type which can be completely handled by the method of successive approximation. However, the easiest and most natural way to handle the present question seems to us to be that based on the use of spherical functions. If we set /168

$$\varrho = \sum_0^\infty \varrho_n, \quad V = \sum_0^\infty V_n, \quad W = \sum_0^\infty W_n,$$

indicating harmonic spherical polynomials of degree n by ϱ_n, V_n, W_n, the right side of equation (62'') will be reduced to

$$\frac{\mu}{\lambda+2\mu}\sum_0^n\Big\{\frac{R}{\mu}\frac{\partial\varrho_n}{\partial x} + \frac{\partial V_n}{\partial y} + \frac{\partial W_n}{\partial z} - \frac{1}{2l}\int_0^l dl\left[y\frac{\partial}{\partial z}\left(\frac{\partial W_n}{\partial y} - \frac{\partial V_n}{\partial z}\right) - \right.$$
$$\left. - z\frac{\partial}{\partial y}\left(\frac{\partial W_n}{\partial y} - \frac{\partial V_n}{\partial z}\right)\right]\Big\}$$

and the quantity under the summation sign will be a harmonic spherical polynomial of degree n-1. Since now ϕ' must be a function harmonic and regular in S, we may assume it, too, to be expanded into a series of harmonic spherical polynomials in order to have

$$\varphi' = \sum_0^\infty \varphi'_n.$$

Since then evidently

$$\frac{1}{2}\varphi' + \frac{5}{2}l\frac{\partial\varphi'}{\partial l} + l^2\frac{\partial^2\varphi'}{dl^2} = \sum_0^n\frac{(2n+1)(n+1)}{2}\varphi'_n$$

the following will have to be true:

$$\frac{(2n+1)(n+1)}{2}\varphi'_n+\frac{\lambda+\mu}{2(\lambda+2\mu)}\frac{\partial^2\varphi'_n}{\partial\beta^2}=\frac{\mu}{\lambda+2\mu}\Big\{\frac{R}{\mu}\frac{\partial \mathfrak{P}_{n+1}}{\partial x}+$$
$$+\frac{\partial V_{n+1}}{\partial y}+\frac{\partial W_{n+1}}{\partial z}-\frac{1}{2l}\int_0^l dl\Big[y\frac{\partial}{\partial z}\Big(\frac{\partial W_{n+1}}{\partial y}-\frac{\partial V_{n+1}}{\partial z}\Big)-$$
$$-z\frac{\partial}{\partial y}\Big(\frac{\partial W_{n+1}}{\partial y}-\frac{\partial V_{n+1}}{\partial z}\Big)\Big]\Big\}.$$

Introducing polar coordinates in place of Cartesian coordinates, it is known that the right side may be given the form /169

$$\frac{\mu}{\lambda+2\mu}\left(\frac{l}{R}\right)^n\sum_0^n{}_i\,(A_{n,i}\cos i\beta+B_{n,i}\,\mathrm{sen}\,i\beta)\,\Theta_{n,i}$$

in which the $\Theta_{n,i}$ are functions of α only, and in which the constants $A_{n,i}$, $B_{n,i}$ must be assumed to be known. If ϕ'_n is put in similar form and the attempt is made to make the last equation identical, it is found that

$$\varphi_n=2\mu\left(\frac{l}{R}\right)^n\sum_0^n{}_i\,\frac{(A_{n,i}\cos i\beta+B_{n,i}\,\mathrm{sen}\,i\beta)\,\Theta_{n,i}}{(2n+1)(n+1)(\lambda+2\mu)-(\lambda+\mu)i^2}$$

must hold and therefore

$$\varphi'=2\mu\sum_1^\infty{}_n\left(\frac{l}{R}\right)^n\sum_0^n{}_i\,\frac{(A_{n,i}\cos i\beta+B_{n,i}\,\mathrm{sen}\,i\beta)\,\Theta_{n,i}}{(2n+1)(n+1)(\lambda+2\mu)-(\lambda+\mu)i^2}. \tag{63}$$

The calculation performed shows at the same time that there is but one function ϕ' satisfying the conditions imposed. Since, moreover, the series

$$\sum_0^\infty{}_n\left(\frac{l}{R}\right)^n\sum_0^n{}_i\,(A_{n,i}\cos i\beta+B_{n,i}\,\mathrm{sen}\,i\beta)\,\Theta_{n,i}$$

is absolutely and uniformly convergent in S, series (63) also has the same property.

When ϕ' has been found, ϕ and θ, as well as $\tilde{\omega}_1$, v,w are consequently found. However, it is easy to show that ψ'_2 and ψ'_3 and therefore ψ_2, ψ_3; $\tilde{\omega}_2$, $\tilde{\omega}_3$;u are determined. The third equation of expression (59) may in fact be written taking the relationship

$$\frac{\partial}{\partial x}(2\psi_1-\psi'_1)+\frac{\partial}{\partial y}(2\psi_2-\psi'_2)+\frac{\partial}{\partial z}(2\psi_3-\psi'_3)=0$$

and equations

$$\tilde{\omega}_2=\frac{1}{4\pi\mu}\frac{\partial \mathfrak{P}}{\partial z}-\frac{1}{4\pi R}\frac{\partial W}{\partial x}-\frac{\lambda+\mu}{4\pi R\mu}\left(z\frac{\partial\varphi}{\partial x}-x\frac{\partial\varphi}{\partial z}\right)+ \Big\} \tag{64}$$

$$+\frac{\lambda+\mu}{4\pi R\mu}l^2\frac{\partial^2\varphi'}{\partial x\partial z}+\frac{\lambda+\mu}{2\pi R\mu}z\frac{\partial\varphi'}{\partial x}-\frac{\lambda+\mu}{4\pi R\mu}x\frac{\partial}{\partial z}(2\varphi-\varphi')+$$
$$+\frac{1}{4\pi R}\left[y\frac{\partial}{\partial x}(2\psi_1-\psi'_1)-x\frac{\partial}{\partial y}(2\psi_1-\psi'_1)\right]. \qquad (64)$$

into consideration, and from a similar formula even $\tilde{\omega}_3$ is found. Quadratures /170 will be used to find ψ_2 and ψ_3, and hence ψ'_2 and ψ'_3. and u.

It is easy to verify the fact that under very broad conditions for the data the formulas found satisfy all the conditions imposed.

4. *Case in which u, M, N are given on the limiting sphere.* Here too we note that, because of the symmetry of the sphere, the problems in which L,v,N, or L,M,w are given on σ do not differ from those proposed here, except in the different name of the coordinate axes. Let us add right away that the difficulties encountered in the solution of this other problem are of the same nature as those which were encountered in the preceding problem. Therefore, we merely allude to its solution.

The starting point will be the formulas

$$\left.\begin{aligned}
u&=\frac{U}{4\pi R}+\frac{\lambda+\mu}{8\pi R\mu}(R^2-l^2)\frac{\partial\varphi}{\partial x},\\
v&=\frac{\mathfrak{M}}{4\pi\mu}-\frac{1}{2\pi R}\left(\frac{\lambda}{2\mu}y\varphi+z\psi_1-x\psi_3\right)+\\
&\qquad+\frac{1}{4\pi R}\left(\frac{\lambda}{2\mu}y\varphi'+z\psi'_1-x\psi'_3\right)+\frac{\lambda+\mu}{4\pi R\mu}l^2\frac{\partial\varphi'}{\partial y}+\\
&\qquad\qquad+\frac{\lambda+\mu}{8\pi R\mu}(R^2-l^2)\frac{\partial\varphi}{\partial y}+\text{cost.}\\
w&=\frac{\mathfrak{N}}{4\pi\mu}-\frac{1}{2\pi R}\left(\frac{\lambda}{2\mu}z\varphi+x\psi_2-y\psi_1\right)+\\
&\qquad+\frac{1}{4\pi R}\left(\frac{\lambda}{2\mu}z\varphi'+x\psi'_2-y\psi'_1\right)+\frac{\lambda+\mu}{4\pi R\mu}l^2\frac{\partial\varphi'}{\partial z}+\\
&\qquad\qquad+\frac{\lambda+\mu}{8\pi R\mu}(R^2-l^2)\frac{\partial\varphi}{\partial z}+\text{cost.}
\end{aligned}\right\} \qquad (65)$$

and an attempt will thus be made to determine θ; $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{\omega}_3$ so that the following /171 equations will be identically satisfied:

$$\begin{aligned}
\theta &= \frac{1}{4\pi R}\frac{\partial U}{\partial x} + \frac{1}{4\pi\mu}\left(\frac{\partial \mathfrak{M}}{\partial y} + \frac{\partial \mathfrak{N}}{\partial z}\right) - \\
&- \frac{\lambda+\mu}{4\pi R\mu}\, l\frac{\partial \varphi}{\partial l} + \frac{\lambda+\mu}{2\pi R\mu}\, l^2\left(\frac{\partial^2 \varphi'}{\partial y^2} + \frac{\partial^2 \varphi'}{\partial z^2}\right) + \frac{\lambda+\mu}{2\pi R\mu}\left(y\frac{\partial \varphi'}{\partial y} + z\frac{\partial \varphi'}{\partial z}\right) - \\
&- \frac{1}{2\pi R}\left[\frac{\lambda}{\mu}\varphi + \frac{\lambda}{2\mu}\left(y\frac{\partial \varphi}{\partial y} + z\frac{\partial \varphi}{\partial z}\right) + z\frac{\partial \psi_1}{\partial y} - x\frac{\partial \psi_3}{\partial y} + x\frac{\partial \psi_2}{\partial z} - y\frac{\partial \psi_1}{\partial z}\right] + \\
&+ \frac{1}{4\pi R}\left[\frac{\lambda}{\mu}\varphi' + \frac{\lambda}{2\mu}\left(y\frac{\partial \varphi'}{\partial y} + z\frac{\partial \varphi'}{\partial z}\right) + z\frac{\partial \psi'_1}{\partial y} - x\frac{\partial \psi'_3}{\partial y} + x\frac{\partial \psi'_2}{\partial z} - y\frac{\partial \psi'_1}{\partial z}\right],
\end{aligned}$$

$$\begin{aligned}
2\varpi_1 &= \frac{1}{4\pi\mu}\left(\frac{\partial \mathfrak{N}}{\partial y} - \frac{\partial \mathfrak{M}}{\partial z}\right) - \\
&- \frac{1}{4\pi R}\left(y\frac{\partial \varphi}{\partial z} - z\frac{\partial \varphi}{\partial y}\right) + \frac{3\lambda+4\mu}{8\pi R\mu}\left(y\frac{\partial \varphi'}{\partial z} - z\frac{\partial \varphi'}{\partial y}\right) - \\
&- \frac{1}{2\pi R}\left[-2\psi_1 + x\left(\frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right) - y\frac{\partial \psi_1}{\partial y} - z\frac{\partial \psi_1}{y z}\right] + \\
&+ \frac{1}{4\pi R}\left[-2\psi'_1 + x\left(\frac{\partial \psi'_2}{\partial y} + \frac{\partial \psi'_3}{\partial z}\right) - y\frac{\partial \psi'_1}{\partial y} - z\frac{\partial \psi'_1}{\partial z}\right],
\end{aligned} \qquad (66)$$

$$\begin{aligned}
2\bar{\omega}_2 &= \frac{1}{4\pi R}\frac{\partial U}{\partial z} - \frac{1}{4\pi\mu}\frac{\partial \mathfrak{N}}{\partial x} - \\
&- \frac{\lambda+\mu}{4\pi R}\left(z\frac{\partial \varphi}{\partial x} - x\frac{\partial \varphi}{\partial z}\right) - \frac{\lambda+\mu}{4\pi R\mu}\, l^2\frac{\partial^2 \varphi'}{\partial x\,\partial z} - \frac{\lambda+\mu}{2\pi R\mu}\, x\frac{\partial \varphi'}{\partial z} + \\
&+ \frac{1}{2\pi R}\left(\frac{\lambda}{2\mu}z\frac{\partial \varphi}{\partial x} + \psi_2 + x\frac{\partial \psi_2}{\partial x} - y\frac{\partial \psi_1}{\partial x}\right) - \\
&- \frac{1}{4\pi R}\left(\frac{\lambda}{2\mu}z\frac{\partial \varphi'}{\partial x} + \psi'_2 + x\frac{\partial \psi'_2}{\partial x} - y\frac{\partial \psi'_1}{\partial x}\right),
\end{aligned}$$

$$2\bar{\omega}_3 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now because /172

$$\begin{aligned}
&-\frac{1}{2\pi R}\left[-2\psi_1 + x\left(\frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right) - y\frac{\partial \psi_1}{\partial y} - z\frac{\partial \psi_1}{\partial z}\right] + \\
&\qquad + \frac{1}{4\pi R}\left[-2\psi'_1 + x\left(\frac{\partial \psi'_2}{\partial y} + \frac{\partial \psi'_3}{\partial z}\right) - y\frac{\partial \psi'_1}{\partial y} - z\frac{\partial \psi'_1}{\partial z}\right] = \\
&\qquad = \frac{1}{4\pi R}\left[4\psi_1 + 2l\frac{\partial \psi_1}{\partial l} - 2\psi'_1 - l\frac{\partial \psi'_1}{\partial l}\right] = \\
&\qquad = \bar{\omega}_1 + \frac{1}{4\pi R}(2\psi_1 - \psi'_1),
\end{aligned}$$

the second equation in expression (66) is written:

$$\left.\begin{aligned}\frac{l}{4\pi R}\frac{\partial}{\partial l}(2\psi_1-\psi'_1)=\frac{1}{4\pi\mu}\left(\frac{\partial\mathfrak{N}}{\partial y}-\frac{\partial\mathfrak{M}}{\partial z}\right)-\\-\frac{1}{4\pi R}\left(y\frac{\partial\varphi}{\partial z}-z\frac{\partial\varphi}{\partial y}\right)+\frac{3\lambda+4\mu}{8\pi R\mu}\left(y\frac{\partial\varphi'}{\partial z}-z\frac{\partial\varphi'}{\partial y}\right);\end{aligned}\right\} \quad (66')$$

while the first equation (66) is easily formulated as

$$\left.\begin{aligned}\theta=\frac{1}{4\pi R}\frac{\partial U}{\partial x}+\frac{1}{4\pi\mu}\left(\frac{\partial\mathfrak{M}}{\partial y}+\frac{\partial\mathfrak{N}}{\partial z}\right)-\frac{2\lambda+\mu}{4\pi R\mu}l\frac{\partial\varphi}{\partial l}+\\+\frac{5\lambda+4\mu}{8\pi R\mu}l\frac{\partial\varphi'}{\partial l}-\frac{\lambda+\mu}{4\pi R\mu}l^2\frac{\partial^2\varphi'}{\partial x^2}+\frac{\lambda+\mu}{2\pi R\mu}x\frac{\partial\varphi}{\partial x}-3\frac{\lambda+\mu}{4\pi R\mu}x\frac{\partial\varphi'}{\partial x}-\\-\frac{\lambda}{4\pi R\mu}(2\varphi-\varphi')-\frac{1}{4\pi R}\left[z\frac{\partial}{\partial y}(2\psi_1-\psi'_1)-y\frac{\partial}{\partial z}(2\psi_1-\psi'_1)\right].\end{aligned}\right\} \quad (66'')$$

Performing the operation $l\frac{\partial}{\partial l}$ on this equation, noting that operations $l\frac{\partial}{\partial l}$ and $z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}$ are exchangeable, and making use of the identity given on page 33, and (66') we find

$$\left.\begin{aligned}2(\lambda+2\mu)l^3\frac{\partial^3\varphi'}{\partial l^3}+(9\lambda+16\mu)l^2\frac{\partial^2\varphi'}{\partial l^2}+(7\lambda+10\mu)l\frac{\partial\varphi'}{\partial l}-\\-2(\lambda+2\mu)l\frac{\partial^3\varphi'}{\partial l\,\partial\beta^2}+(3\lambda+2\mu)\frac{\partial^2\varphi'}{\partial\beta^2}=\\=2l\frac{\partial}{\partial l}\left[\mu\frac{\partial U}{\partial x}+R\left(\frac{\partial\mathfrak{M}}{\partial y}+\frac{\partial\mathfrak{N}}{\partial z}\right)\right]-\\-2R\left[z\frac{\partial}{\partial y}\left(\frac{\partial\mathfrak{N}}{\partial y}-\frac{\partial\mathfrak{M}}{\partial z}\right)-y\frac{\partial}{\partial z}\left(\frac{\partial\mathfrak{N}}{\partial y}-\frac{\partial\mathfrak{M}}{\partial z}\right)\right].\end{aligned}\right\} \quad (67)$$

From this equation ϕ' is found as above by developing the right side into spherical functions; ϕ, θ, and u are then determined, and ψ'_1, ψ_1 and $\tilde{\omega}_1$ by quadratures, starting from expression (66'). Substituting /173

$$\frac{\lambda+2\mu}{2\mu}\frac{\partial\varphi}{\partial z}+\frac{\partial\psi_1}{\partial y},\quad \frac{\lambda+2\mu}{2\mu}\frac{\partial\varphi'}{\partial z}+\frac{\partial\psi'_1}{\partial y},$$

into the third equation (66) for $\frac{\partial\psi_2}{\partial x},\frac{\partial\psi'_2}{\partial x}$, we obtain an equation which, with simple quadratures, can give us ψ'_2, and hence ψ_2, $\tilde{\omega}_2$. In the same manner, we find ψ'_3, ψ_3 and $\tilde{\omega}_3$ and v and w also are consequently found.

IV. Other Boundary Conditions Which Are Suitable for Locating Solutions to Indefinite Equilibrium Equations for Elastic Isotropic Bodies

1. Even a superficial examination of the solutions to problems of elastic equilibrium previously given is enough to convince us of the fact that they already contain all the elements for solving similar problems in which, instead

of stresses or displacements, the values of $\frac{du}{dn}$, $\frac{dv}{dn}$, $\frac{dw}{dn}$ are given, or values of u, v, $\frac{dw}{dn}$, or values of u, $\frac{dv}{dn}$, $\frac{dw}{dn}$ on surface σ, and those which are obtained from these by interchanging the names of the coordinate axes. The general concepts indicated in Section I may serve as guides for attempting to solve these problems in other cases also in which σ differs from a plane or a sphere.

I have no knowledge that research has ever been done on the most general surface conditions which are compatible with indefinite equations (1). However little value such research may have for elasticity theory, I nevertheless believe that it can help illuminate the study of systems of linear partial differential equations of order higher than the first, among the simplest of which must be considered those of isotropic body equilibrium. Here we should like briefly to indicate several forms of surface conditions for which the calculus of variations is sufficient to demonstrate the uniqueness and existence properties. To be sure, these proofs -- besides requiring the fact that u, v, w have regular derivatives in the vicinity -- are subject to the usual criticisms of Dirichlet's principle, but my opinion is that in studies which are general in nature they furnish valuable indications. /174

Let us begin therefore by giving equations

$$\Delta^2 u + \frac{\lambda + \mu}{\mu} \frac{\partial \theta}{\partial x} = 0, \ldots \tag{a}$$

various other forms, availing ourselves of known identities:

$$\Delta^2 u = \frac{\partial \theta}{\partial x} + 2\left(\frac{\partial \varpi_2}{\partial z} - \frac{\partial \varpi_3}{\partial y}\right), \ldots \tag{b}$$

If by means of these identities we eliminate $\Delta^2 u$, $\Delta^2 v$, $\Delta^2 w$ from equations (a), we find the equations (2) already reported:

$$(\lambda + 2\mu) \frac{\partial \theta}{\partial x} + 2\mu \left(\frac{\partial \bar{\omega}_2}{\partial z} - \frac{\partial \bar{\omega}_3}{\partial y}\right) = 0, \ldots \tag{c}$$

If we eliminate $\frac{\partial \theta}{\partial x}$, $\frac{\partial \theta}{\partial y}$, $\frac{\partial \theta}{\partial z}$ instead, equations

$$(\lambda + 2\mu) \Delta^2 u + 2(\lambda + \mu) \left(\frac{\partial \bar{\omega}_3}{\partial y} - \frac{\partial \bar{\omega}_2}{\partial z}\right) = 0, \ldots \tag{d}$$

are found. Finally, adding equations (a) and (b) term by term, we obtain

$$\lambda \frac{\partial \theta}{\partial x} + 2\mu \left(\Delta^2 u + \frac{\partial \bar{\omega}_3}{\partial y} - \frac{\partial \bar{\omega}_2}{\partial z}\right) = 0, \ldots \tag{e}$$

We will pause only on forms (a), (c), (d), (e) of equations (1), but it is clear that others may be obtained from them by the same, or similar, methods.

Let us now multiply equations (a) respectively by δu, δv, δw, let us perform summation, and integrate over a portion of space S. We thus immediately find

$$0 = \int_S dS\,\Sigma\left(\Delta^2 u + \frac{\lambda+\mu}{\mu}\frac{\partial\theta}{\partial x}\right)\delta u =$$
$$= -\int_\sigma d\sigma\,\Sigma\left(\frac{du}{dn} + \frac{\lambda+\mu}{\mu}\theta\cos nx\right)\delta u -$$
$$-\delta\,\frac{1}{2}\int_S\left(\Delta u + \Delta v + \Delta w + \frac{\lambda+\mu}{\mu}\theta^2\right)dS \qquad \text{(f)}$$

where

$$\Delta u = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2, \ldots$$

If in equation (f) we set $\delta u = u$, $\delta v = v$, $\delta w = w$, the same equation is changed into the other /175

$$\frac{1}{2}\int_S\left(\Delta u + \Delta v + \Delta w + \frac{\lambda+\mu}{\mu}\theta^2\right)dS +$$
$$+\int_\sigma d\sigma\,\Sigma\left(\frac{du}{dn} + \frac{\lambda+\mu}{\mu}\theta\cos nx\right)u = 0. \qquad \text{(g)}$$

Assuming that on surface σ the conditions

$$\frac{du}{dn} + \frac{\lambda+\mu}{\mu}\theta\cos nx - c_1 u = 0, \quad \frac{dv}{dn} + \frac{\lambda+\mu}{\mu}\theta\cos ny - c_2 v = 0,$$
$$\frac{dw}{dn} + \frac{\lambda+\mu}{\mu}\theta\cos nz - c_3 w = 0$$

are to be satisfied -- c_1, c_2, c_3 being positive constants -- expression (g) gives us:

$$\frac{1}{2}\int_S\left(\Delta u + \Delta v + \Delta w + \frac{\lambda+\mu}{\mu}\theta^2\right)dS +$$
$$+\int_\sigma (c_1 u^2 + c_2 v^2 + c_3 w^2)\,d\sigma = 0$$

and consequently since $\frac{\lambda + \mu}{\mu} > 0$ the following must be true in all of S and on σ: $u = v = w = 0$. With the usual reasoning, it is then deduced from this result that a system of solutions u, v, w of equations (a) with the boundary conditions

$$L' - c_1 u + \frac{du}{dn} + \frac{\lambda+\mu}{\mu}\theta\cos nx = 0, \ldots \qquad \text{(a')}$$

in which L', M', N' are known functions of the points on σ, is uniquely determined. From expression (f) it results in turn that the functions u, v, w which on σ are subject to the condition

$$\int\left[L' u + M' v + N' w - \frac{1}{2}(c_1 u^2 + c_2 v^2 + c_3 w^2)\right] d\sigma = \text{cost.}$$

and minimize the integral

$$\int_S \left(\Delta u + \Delta v + \Delta w + \frac{\lambda+\mu}{\mu}\theta^2\right) dS$$

satisfy the indefinite equations (a) and the surface conditions (a'). When the existence of this minimum is admitted, the theorem of existence is also proved. The constants c_1, c_2, c_3 may attain the limiting values 0 and ∞ independently of each other. /176

This reasoning may also be applied to other forms (c), (d), (e) of equations (1) and a determinate form of the surface conditions compatible with the same equations will correspond to each of them. These may be written thus

$$L'' - c_1 u + (\lambda + 2\mu)\theta \cos nx + 2\mu(\tilde{\omega}_2 \cos nz - \tilde{\omega}_3 \cos ny) = 0, \ldots \quad (c')$$

$$L''' - c_1 u + (\lambda + 2\mu)\frac{du}{dn} + 2(\lambda + \mu)(\tilde{\omega}_3 \cos ny - \tilde{\omega}_2 \cos nz) = 0, \ldots \quad (d')$$

$$L^{IV} - c_1 u + \lambda\theta \cos nx + 2\mu\left(\frac{du}{dn} + \tilde{\omega}_3 \cos ny - \tilde{\omega}_2 \cos nz\right) = 0, \ldots \quad (e')$$

The latter contains, as a particular case, the form introduced at the beginning which is suitable for giving surface stresses or displacements.

Every time that the surface conditions are such that the one corresponding to the x-axis contains u or $\frac{du}{dn}$ and none of the other displacements or normal derivatives, and a similar situation prevails for those corresponding to the y-axis and z-axis, an attempt can be made to solve the problem of determining the solution of equations (a) corresponding to the given surface conditions using the prinicples expounded in Section I.

V. Some Observations on the Preceding Results

1. The problems relative to a portion of space limited by a plane or by a sphere, which we have studied, must be considered as the simplest among the problems of elastic equilibrium of an isotropic body. These are characterized by the property that the equations, on which determination of the magnitudes θ; $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{\omega}_3$ depends, may be given in the form of ordinary differential equations or partial differential equations, while in the general case, these magnitudes will appear in the equations mentioned as surface integrals independent of them. The feature mentioned is related to the fact that every harmonic function in a portion of space limited by a plane or a sphere is easily expressed by means of only the derivatives of the potential function of a mass distributed on the plane or the sphere with a density proportional to the values which the same harmonic function assumes on the the surface. /177

Let us add that the problems relative to the portion of space limited by a plane, and those relative to that limited by a sphere, are not independent of each other because, on the contrary, the first may be easily derived as particular cases of the second when the radius of the sphere increases indefinitely. However evident it may appear, I do not think that it is useless to demonstrate it rigorously, at least in one case, particularly because I believe that this has never been done explicitly. Therefore, let us assume that the sphere of radius R considered in Section III has its center at point x = 0, y = 0, z = = -R, instead of at the coordinate origin. It will then be necessary to write z + R and ζ + R instead of z and ζ in the formulas in that section. Assuming x, y, z to be finite, it is clear that

$$\lim_{R=\infty}\frac{l}{R}=1,\qquad \lim_{R=\infty}\frac{l^2-R^2}{R}=\lim_{R=\infty}\frac{x^2+y^2+z^2+2Rz}{R}=2z$$

$$\lim_{R=\infty}\frac{l}{R}\frac{\partial}{\partial l}=\lim_{R=\infty}\left(\frac{x}{R}\frac{\partial}{\partial x}+\frac{y}{R}\frac{\partial}{\partial y}+\frac{z}{R}\frac{\partial}{\partial z}+\frac{\partial}{\partial z}\right)=\frac{\partial}{\partial z}.$$

Let us now take into consideration formulas (34) and (35), which give the solution of the problem of elastic equilibrium for the space outside the sphere of radius R when the surface displacements are given:

$$u=\frac{l^2-R^2}{4\pi R}\int_\sigma\frac{u}{r^3}\,d\sigma+\frac{\lambda+\mu}{8\pi R\mu}(l^2-R^2)\frac{\partial}{\partial x}\int_\sigma\frac{\theta}{r}\,d\sigma,\ldots$$

$$\int_\sigma\frac{\theta}{r}\,d\sigma=\frac{\mu}{\lambda+3\mu}\,l^{-\frac{\mu}{\lambda+3\mu}}\int_\infty^l l^{-\frac{\lambda+2\mu}{\lambda+3\mu}}\left\{\frac{\partial}{\partial x}\left[(R^2-l^2)\int_\sigma\frac{u}{r^3}\,d\sigma\right]+\right.$$

$$\left.+\frac{\partial}{\partial y}\left[(R^2-l^2)\int_\sigma\frac{v}{r^3}\,d\sigma\right]+\frac{\partial}{\partial z}\left[(R^2-l^2)\int_\sigma\frac{w}{r^3}\,d\sigma\right]\right\}dl,$$

with

$$\theta=\frac{l^2-R^2}{4\pi R}\int_\sigma\frac{\theta}{r^3}\,d\sigma=\frac{1}{2\pi}\frac{l}{R}\frac{\partial}{\partial l}\int_\sigma\frac{\theta}{r}\,d\sigma+\frac{1}{4\pi R}\int_\sigma\frac{\theta}{r}\,d\sigma.$$

At the limit for R = ∞, the values of u, v, w; θ become: /178

$$u=\frac{z}{2\pi}\int_\sigma\frac{u}{r^3}\,d\sigma+\frac{\lambda+\mu}{4\pi\mu}z\frac{\partial}{\partial x}\int_\sigma\frac{\theta}{r}\,d\sigma,\ldots$$

$$\theta=\frac{z}{2\pi}\int_\sigma\frac{\theta}{r^3}\,d\sigma=\frac{-1}{2\pi}\frac{\partial}{\partial z}\int_\sigma\frac{\theta}{r}\,d\sigma;$$

and since in the case of the sphere

$$\frac{l}{R}\frac{\partial}{\partial l}\int_\sigma\frac{\theta}{r}\,d\sigma=-\frac{\mu^2}{(\lambda+3\mu)^2}\frac{l^{-\frac{\mu}{\lambda+3\mu}}}{R}\int_\infty^l l^{-\frac{\lambda+2\mu}{\lambda+3\mu}}\left\{\frac{\partial}{\partial x}\left[(R^2-l^2)\int_\sigma\frac{u}{r^3}\,d\sigma\right]+\cdots\right\}dl+$$

$$+\frac{\mu}{\lambda+3\mu}\frac{1}{R}\left\{-2x\int_\sigma\frac{u}{r^3}d\sigma-2y\int_\sigma\frac{v}{r^3}d\sigma-2(z+R)\int_\sigma\frac{w}{r^3}d\sigma+\right.$$

$$\left.+(R^2-l^2)\left[\frac{\partial}{\partial x}\int_\sigma\frac{u}{r^3}d\sigma+\frac{\partial}{\partial y}\int_\sigma\frac{v}{r^3}d\sigma+\frac{\partial}{\partial z}\int_\sigma\frac{w}{r^3}d\sigma\right]\right\},$$

at the limit when R = ∞ we have

$$\frac{\partial}{\partial z}\int_\sigma\frac{\theta}{r}d\sigma=-2\pi\zeta=$$

$$=\frac{2\mu}{\lambda+3\mu}\left\{-\int_\sigma\frac{w}{r^3}d\sigma-z\left[\frac{\partial}{\partial x}\int_\sigma\frac{u}{r^3}d\sigma+\frac{\partial}{\partial y}\int_\sigma\frac{v}{r^3}d\sigma+\frac{\partial}{\partial z}\int_\sigma\frac{w}{r^3}d\sigma\right]\right\}=$$

$$=\frac{2\mu}{\lambda+3\mu}\frac{\partial}{\partial z}\left[\frac{\partial}{\partial x}\int_\sigma\frac{u}{r}d\sigma+\frac{\partial}{\partial y}\int_\sigma\frac{v}{r}d\sigma+\frac{\partial}{\partial z}\int_\sigma\frac{w}{r}d\right](^*).$$

These values of u, v, w and θ coincide precisely with those given by expressions (10) and (11) of Section II.

2. It is obvious to note that the same procedure which we have indicated for the three-dimensional case is also true for the two-dimensional case, and even more generally -- if one wished to consider it -- for the case of the equilibrium equations of an isotropic body in linear space with any number of dimensions. /179

In order, however, that one does not feel that I go too far in my estimation of my views -- above all in regard to their originality -- I will now note that it is easier to explain myself by showing sufficiently clearly what these consist of, because the roots of the method which I have followed to obtain the solution of the elastic equilibrium problems of an isotropic body may be found even in the oldest works on the subject. This is particularly true for problems in which certain unknown functions are assumed known and are determined after having satisfied the surface conditions. This is the method that Thomson followed to obtain the solution of the problem of the elastic sphere. The research into those particular solutions which are needed for application of the Betti-Cerruti method also bear traces of these ideas. Even the outstanding solutions in terms of definite integrals of problems of the sphere** and of the half-space given by Professor Almansi have basically this origin. Most explicit of all in following this procedure, however, seems to

* Of course, in this passage to the limit it is assumed that the integrals $\int_\sigma \frac{u}{r} d\sigma$, ... also remain finite in the limit.

** I am happy to cite the work by Professor Somigliana, "On the Equilibrium of an Elastic Body" (Annali della Reale Scuola Normale Superiori di Pisa, 1887) which gives many calculations reminiscent of those of Almansi's, but of which Professor Almansi it is certain had no knowledge.

me to be Professor Cesaro who, in his Introduction to the Mathematical Theory of Elasticity (Introduzione alla teoria matematica dell'Elasticità), after having given the solution of the half-space by the Betti-Cerruti method, gives another which is very similar to the one given in this work and on page 120 says:

"Professor Cerruti has treated the preceding problem 'to give a fairly easy illustration of the general method' proposed by Betti. When one does not have this purpose in mind, but wishes merely to arrive at the solution of the problem of elastic soil, it is very easy by a more rapid and direct procedure to arrive at the general formulas obtained by Professor Cerruti and to do so without giving up 'conduct of the solution in a manner which can provide some light for treating similar problems.' It suffices in fact to take a look at how he notes volume dilation* θ*, then to calculate the displacements* (u, v, w)*, and to deduce from them the expression for* θ*: this function is isolated in a relationship that serves to determine it."* /180

I hope, however, that it will be recognized that with the introduction of formulas (5) and (5') containing the Green functions G and G_1, and with the other observations made in Section I, these ideas come to acquire a generality and a power which they did not have before.

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* The words in single quotes are those of Professor Cerruti, Accademia dei Lincei, 1882, p. 81.